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Rational curves and instantons on the Fano threefold Y_5

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Abstract

This thesis is an investigation of the moduli spaces of instanton bundles on the Fano threefold Y_5 (a linear section of $\mathrm{Gr}(2, 5)$). It contains new proofs of classical facts about lines, conics and cubics on Y_5 , and about linear sections of Y_5 .

The main original results are a Grauert–Mülich theorem for the splitting type of instantons on conics, a bound to the splitting type of instantons on lines and an SL_2 -equivariant description of the moduli space in charge 2 and 3.

Using these results we prove the existence of a unique SL_2 -equivariant instanton of minimal charge and we show that for all instantons of charge 2 the divisor of jumping lines is smooth. In charge 3, we provide examples of instantons with reducible divisor of jumping lines. Finally, we construct a natural compactification for the moduli space of instantons of charge 3, together with a small resolution of singularities for it.

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Introduction

The goal of this thesis is an investigation of instanton vector bundles on a Fano threefold of index 2 and degree 5, a natural generalization of instanton vector bundles on \mathbb{P}^3 .

Definition. An instanton on \mathbb{P}^3 is a μ -stable vector bundle E of rank 2 such that $c_1(E) = 0$ and $H^1(E(-2)) = 0$. Its second Chern class $c_2(E)$ is called the *charge* of E .

The standard approach [AHDM78] (see also [OSS80]) to the description of instantons on \mathbb{P}^3 and of their moduli space is via the Beilinson spectral sequence. Using this approach one can write any instanton E of charge n as the middle cohomology of a complex

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus n} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 2n+2} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus n} \rightarrow 0 \quad (1)$$

such that the first map is fiberwise injective and the last map is surjective. Such a complex is usually called a *monad* for E and is determined by E up to the action of a group. This leads to a description of the moduli space of instantons as a GIT quotient. This description turns out to be too complicated for large n to provide answers to the natural questions about the moduli space of instantons: smoothness, irreducibility and rationality. In the end, some of these questions were solved in recent works of Verbitsky, Jardim, Markushevitch and Tikhomirov by a different technique (see [JV11], [MT10], [Tik12]).

On the other hand, for small values of n one can give a more explicit description of the moduli space of instantons (see [ES81]) by analyzing the loci of jumping lines.

Definition. A line L is jumping for an instanton E if E restricted to L is not trivial. We will say that a line L is k -jumping for E if E restricted to L is isomorphic to $\mathcal{O}_L(-k) \oplus \mathcal{O}_L(k)$.

By Grauert–Mülich theorem the locus of jumping lines for an instanton E is a hypersurface S_E of degree n in the Grassmannian $\mathrm{Gr}(2, 4)$ parameterizing lines in \mathbb{P}^3 . It comes with a rank one torsion free sheaf \mathcal{L}_E and one can show that an instanton E can be reconstructed from the pair (S_E, \mathcal{L}_E) . However, it turns out to be quite hard to characterize which pairs (S_E, \mathcal{L}_E) give rise to instantons, at least for higher values of the charge. One of the reasons for this is the high dimension of S_E and its singularities.

Recently, the definition of instanton bundle was extended from \mathbb{P}^3 to other Fano threefolds of Picard number 1.

Definition ([Fae11]). A rank 2 vector bundle E with $c_1(E) = 0$ or 1 on a Fano threefold X is an instanton if there is a twist $E \otimes \mathcal{O}_X(t) = F$ such that

$$F \cong F^* \otimes \omega_X, \quad H^1(X, F) = 0$$

Faenzi proved some general results about instantons on Fano threefolds, such as the existence of a good component of arbitrary charge (see [Fae11]). On the other hand, Kuznetsov investigated in greater detail the case of Fano threefolds of index 2. In

particular, in [Kuz12] the locus of jumping lines D_E was defined together with a theta-sheaf \mathcal{L}_E on it. Moreover, motivated by the Grauert–Mülich theorem, the following conjecture was suggested.

Conjecture 1 ([Kuz12]). For an instanton E on a Fano threefold of index 2 the generic line is not jumping. An instanton E can be reconstructed from its theta-sheaf \mathcal{L}_E .

In the case of Fano threefolds of index 2 and degree 4 or 5, [Kuz12] provides more concrete descriptions of the moduli spaces of instantons.

Namely, in the case of the degree 4 threefold the moduli spaces of instantons were shown to be related to the moduli spaces of self dual vector bundles on a curve of genus 2. Using this, the first part of conjecture 1 was reinterpreted and the second part was proved.

The degree 5 case is that of a triple linear section of $\mathbb{G}r(2, 5)$.

Definition. The variety Y is the transverse intersection of the Plücker embedding $\mathbb{G}r(2, 5) \subset \mathbb{P}^9$ with a \mathbb{P}^6 .

It turns out that Y is similar to \mathbb{P}^3 in many respects. For example, both admit an action of SL_2 with an open orbit and both admit a full exceptional collection for their derived category.

Again in [Kuz12], these similarities with \mathbb{P}^3 were used to show that every instanton on Y can be represented as the middle cohomology of the monad

$$0 \rightarrow \mathcal{U}^{\oplus n} \rightarrow \mathcal{O}_Y^{\oplus 4n+2} \rightarrow \mathcal{U}^{*\oplus n} \rightarrow 0,$$

where \mathcal{U} is the restriction of the tautological bundle to Y , in analogy with (1). As a corollary a GIT description of the moduli space of instantons via special nets of quadrics was found, a reinterpretation of the first part of conjecture 1 was given and the second part was proved.

The goal of this thesis is an extension of the results of [Kuz12] on moduli spaces of instantons on Y and a detailed study of moduli spaces of small charge, namely charge 2 (minimal instantons) and charge 3.

We start with an exposition of various facts about the geometry of Y , some of which are known to experts. The references for these facts are scattered in the literature and some of them do not explicitly refer to the SL_2 -structure, so that we prefer providing complete proofs for most of them. In particular, we describe several useful exceptional collection in $D^b(Y)$, the bounded derived category of coherent sheaves on Y , show that the Hilbert scheme of lines on Y identifies with \mathbb{P}^2 and that the Hilbert scheme of conics on Y identifies with \mathbb{P}^4 . We also write down explicit SL_2 -equivariant resolutions for the structure sheaves of lines and conics, discuss the natural stratification of the Hilbert scheme of conics and describe the natural incidence correspondences on the products of these Hilbert schemes. Finally, we apply the same methods to cubic curves in Y . In all these constructions we pay special attention to the induced action of SL_2 .

In section 3 we introduce the following construction of Y : we show that the blowup of the Veronese surface in \mathbb{P}^4 identifies naturally with the projectivization of the vector bundle \mathcal{U} on Y . This description turns out to be very useful later, when we discuss the geometry of hyperplane sections of Y and their relation to the Grassmannian of planes in \mathbb{P}^4 . Again, all the results of this section are SL_2 -equivariant.

In section 4 we start discussing instantons on Y . We remind the necessary definitions and constructions from [Fae11] and [Kuz12], in particular the monadic description and some results about jumping lines. We state here the first new result of the thesis, which will be of great use in the description of the moduli space of minimal instantons.

Theorem (4.17, 4.18). *If E is an instanton of charge n then E has no k -jumping lines for $k \geq n$.*

A direct consequence of theorem 4.17 is the following proposition providing an explicit description of instantonic theta-characteristics of degree 2.

Proposition (5.2). *If E is an instanton of charge 2 then the curve of jumping lines D_E is a smooth conic and the sheaf \mathcal{L}_E is isomorphic to $\mathcal{O}_{D_E}(-1)$. In particular, conjecture 1 holds true for instantons of charge 2.*

We also introduce the notion of a jumping conic for an instanton. Of course one can just say that a conic C is jumping for E if the restriction of E to C is nontrivial, but it is not clear how to encode nontriviality of E on C in a sheaf on the Hilbert space of conics. We show that a reducible conic $C = L_1 \cup L_2$ (resp. nonreduced conic $C = 2L$) is a jumping conic for an instanton E if and only if the restriction of E to either of the lines L_1 and L_2 (resp. to L) is nontrivial. Moreover, we show that

Proposition (4.22). *If E is an instanton and C is a conic then C is a jumping conic if and only if $H^i(C, E \otimes \mathcal{U}|_C) \neq 0$ for some i .*

By means of proposition 4.22, we also show that an analogue of the Grauert–Mülich theorem holds for conics.

Theorem (4.24, 4.29). *For an instanton E of charge n the scheme of jumping conics of E is a divisor of degree n in \mathbb{P}^4 which comes with a natural sheaf of rank 2. In particular, the generic conic is non-jumping.*

So, in some sense, conics behave better than lines, but as the dimension of the space of conics is higher, the information we get is harder to use.

In section 5 we discuss in detail the moduli space of minimal instantons. This moduli space is also known in the literature as the moduli space of aCM bundles (see [BF09]). We show that the moduli space \mathcal{MI}_2 identifies with an explicit open subset of \mathbb{P}^5 .

Theorem (5.8). *The moduli space \mathcal{MI}_2 of charge 2 instantons on Y can be represented as*

$$\mathcal{MI}_2 \cong \mathbb{P}(S^2\mathbb{C}^3) \setminus (\Delta \cup H),$$

where Δ is the symmetric determinantal cubic and H is the unique SL_2 -invariant hyperplane.

A consequence of the description of theorem 5.8 is the following.

Corollary (5.9). *There is a unique instanton E_0 of charge 2 on Y with an SL_2 -equivariant structure.*

In section 6 we consider the moduli space of instantons of charge 3. We construct a natural map $\beta : \mathcal{MI}_3 \rightarrow B := \mathbb{Gr}(3, 5)$ and show that its flattening stratification on B consists of only two strata: the closed stratum $B^s \subset B$ identifies with the image of the projective plane parameterizing lines on Y under a special embedding $\kappa : \mathbb{P}^2 \rightarrow B$ constructed in section 3, and the open stratum $B^n := B \setminus B^s$ is its complement.

After constructing β , we also construct a natural compactification $\overline{\mathcal{MI}}_3$ of \mathcal{MI}_3 and describe it.

Theorem (6.23). *There is a natural compactification $\overline{\mathcal{MI}}_3$ of \mathcal{MI}_3 such that the map $\beta : \mathcal{MI}_3 \rightarrow B$ extends to a regular map $\tilde{\beta} : \overline{\mathcal{MI}}_3 \rightarrow B$ which is a \mathbb{P}^3 -fibration over B^n and a $\mathbb{Gr}(2, 5)$ -fibration over B^s .*

We show that instantons E such that $\beta(E) \in B^s$ form a divisorial family and have many special properties (because of this we call them *special*). One of the characterizing properties is the following

Proposition (6.24). *An instanton E is special if and only if it has a 2-jumping line L . For a special instanton E there is a unique 2-jumping line L and $\kappa(L) = \beta(E)$.*

In section 7, the results of section 6 are used to show how singular the theta-sheaf \mathcal{L}_E of jumping lines can be for an instanton E of charge 3. While in the case of minimal instantons the theta-characteristics admit the simple description of corollary 5.2, for $c_2(E) = 3$ we prove the following propositions

Proposition (7.2). *The theta-sheaf of E is not locally free if and only if E is special.*

Proposition (7.3). *If the plane $\beta(E) \subset \mathbb{P}^4$ is a generic tangent plane to the Veronese surface in \mathbb{P}^4 , then the theta-sheaf of E is supported on a reducible curve.*

It turns out that the compactified moduli space $\overline{\mathcal{MI}}_3$ is singular on a closed subset of codimension 3 which is contained in $\overline{\mathcal{MI}}_3 \setminus \mathcal{MI}_3$ and in $\beta^{-1}(B^s)$.

The last section is devoted to the construction and the description of an explicit desingularization

$$\tilde{\pi} : \widetilde{\mathcal{MI}}_3 \rightarrow \overline{\mathcal{MI}}_3$$

We construct $\widetilde{\mathcal{MI}}_3$ by blowing up the (non-Cartier) divisor of special instantons inside $\overline{\mathcal{MI}}_3$ and we identify $\widetilde{\mathcal{MI}}_3$ with a \mathbb{P}^3 -fibration over the blowup \tilde{B} of B in B^s .

Theorem (8.4). *There is a small contraction*

$$\tilde{\pi}_4 : \widetilde{\mathcal{MI}}_3 \rightarrow \overline{\mathcal{MI}}_3$$

which is a resolution of the singularity of $\overline{\mathcal{MI}}_3$. Moreover, the pushforward of the structure sheaf of $\widetilde{\mathcal{MI}}_3$ is the structure sheaf of \mathcal{MI}_3 .

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1 Some preliminary facts

We work over \mathbb{C} . All schemes are assumed to be Noetherian and separated. In this section we set the notation which we will use for derived categories and we state a few more results (Beilinson spectral sequence, Grauert–Mülich theorem, Hoppe’s criterion) to which we will refer in the rest of the thesis.

1.1 Semi-orthogonal decompositions

In any triangulated category \mathcal{T} the shift is denoted by $[1]$. For any two objects F, G of \mathcal{T} , the space $\mathrm{Hom}(F, G[p])$ is denoted by $\mathrm{Ext}^p(F, G)$ and the cohomology of $\mathrm{RHom}(F, G)$, that is to say $\bigoplus_p \mathrm{Ext}^p(F, G)[-p]$, is denoted by $\mathrm{Ext}^\bullet(F, G)$.

We work with triangulated categories \mathcal{T} which are Ext -finite, that is to say that for any pair of objects F, G the graded vector space $\mathrm{Ext}^\bullet(F, G)$ is finite dimensional. In other words, $\mathrm{Hom}(F, G)$ is always finite dimensional and $\mathrm{Hom}(F, G[i])$ vanishes for almost all integers $i \in \mathbb{Z}$.

We usually think of $\mathrm{Ext}^\bullet(F, G)$ as a non-symmetric scalar product with values in the category of graded finite dimensional vector spaces. When a subcategory $\mathcal{A} \subset \mathcal{T}$ is given, the subcategory

$$\mathcal{A}^\perp = \{T \in \mathcal{T} \text{ s.t. } \forall A \in \mathcal{A} \quad \mathrm{Ext}^\bullet(A, T) = 0\}$$

is called the right orthogonal of \mathcal{A} in \mathcal{T} . Respectively, the subcategory

$${}^\perp\mathcal{A} = \{T \in \mathcal{T} \text{ s.t. } \forall A \in \mathcal{A} \quad \mathrm{Ext}^\bullet(T, A) = 0\}$$

is called the left orthogonal of \mathcal{A} in \mathcal{T} .

Definition 1.1. A full triangulated subcategory \mathcal{A} of \mathcal{T} is admissible if the inclusion functor has a left and a right adjoint.

If the inclusion functor of \mathcal{A} has only a left or right adjoint, the category \mathcal{A} is called respectively left or right admissible. By [Kuz07, sec. 2.2/2.3] (which summarizes results and definitions from [Bon90] [BK90] [BVdB03]), in the case \mathcal{T} is the derived category of a smooth variety, the following theorem holds.

Theorem 1.2 ([BK90]). *For X smooth projective variety over a field k , any left or right admissible subcategory of $D^b(X)$ is admissible.*

Definition 1.3. A semiorthogonal decomposition of a triangulated category \mathcal{T} is an ordered set of full triangulated subcategories $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ such that the following two conditions hold:

- For all $i > j$ and all $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$

$$\mathrm{Hom}(A_i, A_j) = 0$$

- For any object $T \in \mathcal{T}$ there exist objects T_i and maps f_i

$$0 = T_n \xrightarrow{f_n} T_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 = T$$

such that the cones of $f_i : T_i \rightarrow T_{i-1}$ lie in \mathcal{A}_i .

A sequence of maps as in definition 1.3 is usually called a filtration for T .

Definition 1.4. Given an object T in \mathcal{T} , a filtration for T is a set of objects T_i and maps f_i such that

$$0 \cong T_m \xrightarrow{f_m} T_{m-1} \xrightarrow{f_{m-1}} \dots \xrightarrow{f_{n+2}} T_{n+1} \xrightarrow{f_{n+1}} T_n \cong T \quad (2)$$

If we denote by $L^i[-i]$ the cone of f_{i+1} , there is an induced sequence in the opposite direction

$$0 \rightarrow L^n \xrightarrow{g^n} L^{n+1} \xrightarrow{g^{n+1}} \dots \xrightarrow{g^{m-2}} L^{m-1} \rightarrow 0 \quad (3)$$

such that $g^{i+1} \circ g^i = 0$. Sequences (3) and (2) fit in a diagram

$$\begin{array}{ccccccc} 0 & \xrightarrow{f_m} & T_{m-1} & \xrightarrow{f_{m-1}} & \dots & \xrightarrow{f_{n+2}} & T_{n+1} \xrightarrow{f_{n+1}} T \\ & \searrow \text{dashed} & \swarrow & \searrow \text{dashed} & \swarrow & \searrow \text{dashed} & \swarrow \\ & L^{m-1}[-m+1] & \xleftarrow{g^{m-2}} & L^{m-2}[-m+2] & \dots & L^{n+1}[-n-1] & \xleftarrow{g^n} L^n[-n] \end{array} \quad (4)$$

where all bottom triangles are commutative and those marked with \star are distinguished. Dashed arrows denote maps of degree one. The diagram (4) is usually called Postnikov tower (or Postnikov system). One also usually says that T is the convolution of the Postnikov tower (4).

The following is a simple and classical fact, but it is fundamental.

Theorem 1.5 ([Bon90]). *The Postnikov tower induced by a semi-orthogonal decomposition is functorial.*

Proof. By definition of semi-orthogonal decomposition it is enough to prove the theorem for a decomposition $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ and then proceed by induction.

Choose a filtration for each object T in \mathcal{T} . Given a map $\varphi : S \rightarrow T$ we will show that it extends uniquely to a map of distinguished triangles

$$\begin{array}{ccccccc} S_1 & \longrightarrow & S & \longrightarrow & L & \longrightarrow & S_1[1] \\ \downarrow \varphi_1 & & \downarrow \varphi & & \downarrow \psi & & \downarrow \varphi_1[1] \\ T_1 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & T_1[1] \end{array}$$

The fact that ψ commutes with both φ and $\varphi_1[1]$ comes from the axiom of triangulated category which says that there is at least one such ψ .

Functoriality follows from uniqueness of φ_1 and ψ . □

An object $E \in \mathcal{A}$ is called exceptional if $\text{Ext}^\bullet(E, E) = \mathbb{C}$. A set of exceptional objects E_1, \dots, E_n is called an exceptional collection if $\text{Ext}^k(E_i, E_j) = 0$ for $i > j$. If moreover $\text{Ext}^k(E_i, E_j) = 0$ whenever $k \neq 0$ the exceptional collection is called a strong exceptional collection.

A way to induce semiorthogonal decompositions is to look for exceptional objects in \mathcal{T} , as the following theorem shows. Note that, when included in a semiorthogonal decomposition, E_i stands for the category generated by E_i .

Theorem 1.6. *Given an exceptional collection E_1, \dots, E_n in \mathcal{T} , there are induced semiorthogonal decompositions*

$$\langle E_1, \dots, E_n, {}^\perp \langle E_1, \dots, E_n \rangle \rangle$$

and

$$\langle \langle E_1, \dots, E_n \rangle^\perp, E_1, \dots, E_n \rangle$$

If the left or the right orthogonal of $\langle E_1, \dots, E_n \rangle$ (i.e. the minimal triangulated subcategory containing E_1, \dots, E_n) is 0, the collection is called a full exceptional collection. In this case both the left and the right orthogonal to $\langle E_1, \dots, E_n \rangle$ vanish.

Given a semiorthogonal decomposition, one can obtain other semiorthogonal decompositions via functors which are called left and right mutations.

Let $j : \mathcal{A} \rightarrow \mathcal{T}$ be an admissible subcategory, so that $\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$ and $\langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$ are semiorthogonal decompositions. Denote the left and right adjoint functors to j by j^* and $j^!$. Denote the embedding functors of ${}^\perp \mathcal{A}$ and \mathcal{A}^\perp in \mathcal{T} by l and r , and their (respectively right and left) adjoint functors by $l^!$ and r^* . It is easy to check that for $T \in \mathcal{T}$ we have

$$l^!(T) = \text{Cone}(T \rightarrow jj^*T)[-1]$$

and

$$r^*(T) = \text{Cone}(jj^!T \rightarrow T)$$

Definition 1.7. The functor $ll^! : \mathcal{T} \rightarrow \mathcal{T}$ is called right mutation through \mathcal{A} and is denoted by $\mathbb{R}_{\mathcal{A}}$. The functor $rr^*[-1] : \mathcal{T} \rightarrow \mathcal{T}$ is called left mutation through \mathcal{A} and is denoted by $\mathbb{L}_{\mathcal{A}}$.

To spell out definition 1.7, we point out that the left and right mutation functors are characterized by the fact that there are distinguished triangles

$$\mathbb{L}_{\mathcal{A}}(T) \longrightarrow jj^!T \longrightarrow T \longrightarrow \mathbb{L}_{\mathcal{A}}(T)[1]$$

and

$$\mathbb{R}_{\mathcal{A}}(T)[-1] \longrightarrow T \longrightarrow jj^*T \longrightarrow \mathbb{R}_{\mathcal{A}}(T)$$

The functors $\mathbb{R}_{\mathcal{A}}$ and $\mathbb{L}_{\mathcal{A}}$ return zero on \mathcal{A} , but they are isomorphisms when restricted to the (respectively right and left) orthogonals. More precisely

$$l^!r = l^!\mathbb{R}_{\mathcal{A}}r : \mathcal{A}^\perp \rightarrow {}^\perp \mathcal{A}$$

and

$$r^*l = r^*\mathbb{L}_{\mathcal{A}}l : {}^\perp\mathcal{A} \rightarrow \mathcal{A}^\perp$$

are equivalences of triangulated categories as it is easy to check that the functor $l^! \cdot r \cdot r^* \cdot l$ is equivalent to the identity of ${}^\perp\mathcal{A}$ and that the functor $r^* \cdot l \cdot l^! \cdot r$ is equivalent to the identity of \mathcal{A}^\perp .

A special case of the above setting is when \mathcal{A} is $D^b(k)$, with k a field. In this case j sends k to an exceptional object E , and the functors j^* and $j^!$ are respectively $\text{Ext}^\bullet(-, E)^*$ and $\text{Ext}^\bullet(E, -)$. In this case the unit and the counit of the adjunction are respectively the evaluation and coevaluation maps for E , i.e. the universal

$$\text{Ext}^\bullet(E, T) \otimes E \rightarrow T$$

and

$$T \rightarrow \text{Ext}^\bullet(T, E)^* \otimes E$$

Now, let $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be a semiorthogonal decomposition into admissible subcategories. Denote the collection $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ by \mathfrak{A} ; we are going to construct two collections which we will denote by $\mathbb{R}_i\mathfrak{A}$ and $\mathbb{L}_i\mathfrak{A}$.

Definition 1.8 (Lemma 2.3 [Kuz06]). Given $\mathfrak{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ semiorthogonal decomposition, assume that each \mathcal{A}_i is admissible. Then for each $i \in [1, n-1]$ there is a semiorthogonal decomposition

$$(\mathbb{R}_i\mathfrak{A})_j = \begin{cases} \mathcal{A}_j & \text{if } j \neq i, i+1 \\ \mathcal{A}_{i+1} & \text{if } j = i \\ \mathbb{R}_{\mathcal{A}_{i+1}}\mathcal{A}_i = {}^\perp\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1} \rangle \cap \langle \mathcal{A}_{i+2}, \dots, \mathcal{A}_n \rangle^\perp & \text{if } j = i+1. \end{cases}$$

called the i -th right mutation of \mathfrak{A} and a semi-orthogonal decomposition

$$(\mathbb{L}_i\mathfrak{A})_j = \begin{cases} \mathcal{A}_j & \text{if } j \neq i, i+1 \\ \mathbb{L}_{\mathcal{A}_i}\mathcal{A}_{i+1} = {}^\perp\langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1} \rangle \cap \langle \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_n \rangle^\perp & \text{if } j = i \\ \mathcal{A}_i & \text{if } j = i+1 \end{cases}$$

called the i -th left mutation of \mathfrak{A} .

We are now going to introduce two semiorthogonal decompositions related to \mathfrak{A} : they are called the left dual and the right dual to \mathfrak{A} due to the vanishing of the entries of the matrices $\text{RHom}({}^\vee\mathcal{A}_i, \mathcal{A}_j)$ and $\text{RHom}(\mathcal{A}_i, {}^\vee\mathcal{A}_j)$ such that $i+j \neq n+1$.

Definition 1.9. The right dual decomposition of \mathfrak{A} is

$$\mathfrak{A}^\vee = (\mathbb{L}_1 \dots \mathbb{L}_{n-1}) \circ \dots \circ (\mathbb{L}_1 \mathbb{L}_2) \circ (\mathbb{L}_1)(\mathfrak{A})$$

The left dual decomposition of \mathfrak{A} is

$${}^\vee\mathfrak{A} = (\mathbb{R}_{n-1} \dots \mathbb{R}_1) \circ \dots \circ (\mathbb{R}_{n-1} \mathbb{R}_{n-2}) \circ (\mathbb{R}_{n-1})(\mathfrak{A})$$

In the case the categories \mathcal{A}_i are generated by an exceptional element, one can refine the idea of dual decomposition to that of dual collection.

Definition 1.10. Let $\mathcal{T} = \langle E_1, \dots, E_n \rangle$ be a full exceptional collection for \mathcal{T} . Set

$$E_i^\vee = \mathbb{L}_{E_1} \mathbb{L}_{E_2} \cdots \mathbb{L}_{E_{n-1}} E_{n-i+1}$$

and

$${}^\vee E_i = \mathbb{R}_{E_n} \mathbb{R}_{E_{n-1}} \cdots \mathbb{R}_{E_{n-i+2}} E_{n-i+1}$$

The full exceptional collections $\langle E_n^\vee, \dots, E_1^\vee \rangle$ and $\langle {}^\vee E_n, \dots, {}^\vee E_1 \rangle$ are called respectively right dual collection and left dual collection.

With the above choices, the right and left dual collections are characterized by the following property (see [GK04, sec.2.6]).

$$\mathrm{Ext}^k({}^\vee E_i, E_j) = \mathrm{Ext}^k(E_i, E_j^\vee) = \begin{cases} \mathbb{C} & \text{if } i + j = n + 1 \text{ and } i = k + 1 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

1.2 Beilinson spectral sequence

Given a filtration for an object A in $D^b(X)$, one can construct a spectral sequence analogous to the one induced by a filtered complex. Recall that the filtration induces a Postnikov tower whose convolution is A . In the notation of diagram (4), we have the following proposition.

Proposition 1.11 (e.g. [GM03] ex. IV.2.2). *Given a filtration T_i, f_i for an object A in $D^b(X)$, there is a natural spectral sequence*

$$E_1^{p,q} = H^q(L^p)$$

with differential $d^p = H^q(g^p)$ and converging to $H^{p+q}(A)$.

Proof. Replace the L^i to reduce to the case where f_i are actual morphisms of complexes, then use the double complex spectral sequence. \square

As we will often use proposition 1.11, let us clarify in the following remarks what we mean by natural and what the direction of differentials in the spectral sequence is.

Remark 1.12. By naturality of the induced spectral sequence we mean that given two Postnikov towers whose convolution is A_1 and A_2 , and given a map between the two towers (i.e. a collection of maps between the objects in the tower commuting with all maps in the tower), then there is an induced map of spectral sequences. Moreover, such maps of spectral sequences depend functorially on the maps between Postnikov towers.

Remark 1.13. The spectral sequence can be thought of as the spectral sequence induced by a bicomplex whose columns are L^i and whose horizontal arrows are g^i . Entries $E_1^{p,q}$ with $p \notin [n, m-1]$ vanish.

$$\begin{array}{ccccccc}
H^{j+1}(L^n) & \xrightarrow{H^{j+1}(g^n)} & H^{j+1}(L^{n+1}) & \xrightarrow{H^{j+1}(g^{n+1})} & \cdots & \xrightarrow{H^{j+1}(g^{m-2})} & H^{j+1}(L^{m-1}) \\
& \searrow d_2 & & & & & \\
H^j(L^n) & \xrightarrow{H^j(g^n)} & H^j(L^{n+1}) & \xrightarrow{H^j(g^{n+1})} & \cdots & \xrightarrow{H^j(g^{m-2})} & H^j(L^{m-1}) \\
& \searrow d_2 & & & & & \\
H^{j-1}(L^n) & \xrightarrow{H^{j-1}(g^n)} & H^{j-1}(L^{n+1}) & \xrightarrow{H^{j-1}(g^{n+1})} & \cdots & \xrightarrow{H^{j-1}(g^{m-2})} & H^{j-1}(L^{m-1})
\end{array}$$

It follows that the last non trivial differential is at most at page E_{n-m+1} .

A common situation where a filtration for an object in $D^b(X)$ is available is when there is a semi-orthogonal decomposition of $D^b(X)$. The archetype of such a situation is $X = \mathbb{P}^n$: its derived category admits several full exceptional collections, the most notable being $\langle \mathcal{O}_{\mathbb{P}^n}(-n), \dots, \mathcal{O}_{\mathbb{P}^n} \rangle$. Together with its dual collection, it gives rise to a resolution of the diagonal inside $\mathbb{P}^n \times \mathbb{P}^n$.

The point in having a resolution for the diagonal of a scheme X is that one can always rewrite the identity functor of $D^b(X)$ as the Fourier–Mukai transform with kernel the structure sheaf of the diagonal. As on \mathbb{P}^n there is such a resolution, one can take any object A in $D^b(\mathbb{P}^n)$, pull it back to $\mathbb{P}^n \times \mathbb{P}^n$, tensor it by the resolution of the diagonal, push it forward to the other copy of \mathbb{P}^n and finally recover the initial A . The Grothendieck spectral sequence is in this case a spectral sequence converging to $H^\bullet(A)$, which is known as the Beilinson spectral sequence.

Theorem 1.14 ([OSS80]). *For any A in $D^b(\mathbb{P}^n)$ there are spectral sequences*

$$'E_1^{pq} = H^q(A \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}(p)$$

and

$$''E_1^{pq} = H^q(A(p)) \otimes \Omega^{-p}(-p)$$

converging to $H^{p+q}(A)$.

The existence of a Beilinson-type spectral sequence on a variety X is a more general fact, depending only on the choice of a full exceptional collection on X . As we will need to decompose families of sheaves with respect to a full exceptional collection, we will state a relative version of the Beilinson spectral sequence.

Assume a scheme X has a full exceptional collection E_1, \dots, E_n . Then there is an induced left dual collection ${}^\vee E_1, \dots, {}^\vee E_n$ and a full exceptional collection for $X \times X$

$$D^b(X \times X) = \langle E_i \boxtimes ({}^\vee E_j)^* \rangle_{i=1 \dots n, j=1 \dots n}$$

The filtration of the diagonal of X with respect to the above exceptional collection takes a very simple form.

Theorem 1.15 ([Kuz09]). *The structure sheaf of the diagonal $\Delta_* \mathcal{O}_X$ has a filtration whose i -th shifted successive cone L^i is $E_{i+1} \boxtimes (\vee E_{n-i})^*[n-1]$.*

Remark 1.16. The statement in [Kuz09, Prop. 3.8] only says that the subquotients belong to the subcategory generated by $E_{i+1} \boxtimes (\vee E_{n-i})^*$. To check that the cones are the ones in the statement of theorem 1.15, use the filtration of the diagonal to induce a filtration of E_i , and afterwards use the spectral sequence of proposition 1.11 on the filtration for E_i .

One of the consequences of theorem 1.15 is the following result, which can be seen as an analogue of the Beilinson spectral sequence in a category where a full exceptional decomposition is available.

Theorem 1.17. *Let X be a smooth projective variety and let $D^b(X)$ be generated by an exceptional sequence of pure objects E_1, \dots, E_n . Then for any A in $D^b(X)$ there is a spectral sequence*

$$E_1^{p,q} = \text{Ext}^{n+q-1}(\vee E_{n-p}, A) \otimes E_{p+1}$$

converging to $H^{p+q}(A)$.

The statement and its proof can be found both in [GK04, sec 2.7.3] and in [Rud90, cor. 3.3.2]. A more general form can be found in [Böh06, thm 2.1.14].

With theorem 1.17 in mind, we write a spectral sequence for a family of objects in $D^b(X)$ parametrized by a scheme S . First, choose a notation for the projections between products of X and S

$$\begin{array}{ccc} X \times X \times S & \xrightarrow{p_2} & X \times S \\ p_1 \downarrow & & \downarrow q \\ X \times S & \xrightarrow{q} & S \end{array}$$

Next, construct the diagonal of $X \times X \times S$ relative to S , namely $\Delta_X \times S$. Finally, by pulling back the filtration for the diagonal of $X \times X$ to $X \times X \times S$, we obtain a filtration for $\Delta_X \times S$ whose successive cones are $E_i \boxtimes (\vee E_{n-i+1})^* \boxtimes \mathcal{O}_S$.

Theorem 1.18. *Let X be a projective variety having a full exceptional collection. Let S be any scheme. For any $\mathcal{F} \in D^b(X \times S)$ there is a spectral sequence*

$$E_1^{p,q} = E_{p+1} \boxtimes R^{n+q-1} q_* R\mathcal{H}om_{X \times S}(\vee E_{n-p} \boxtimes \mathcal{O}_S, \mathcal{F}) \quad (6)$$

converging to $H^{p+q}(\mathcal{F})$ which is functorial in \mathcal{F} .

Proof. The main step in the proof is rewriting the identity functor of $D^b(X \times S)$ as the composition of the following functors: first pullback via p_2 , then tensor by the structure sheaf of the relative diagonal $\Delta_X \times S$, finally pushforward via p_1 . This is the same as saying that the identity of $D^b(X \times S)$ is the Fourier–Mukai transform with kernel the structure sheaf of $\Delta_X \times S$.

Pullback the filtration for \mathcal{O}_{Δ_X} of theorem 1.15 to get a filtration of the relative diagonal $\mathcal{O}_{\Delta_X \times S}$ with successive cones $E_{i+1} \boxtimes (\vee E_{n-i})^* \boxtimes \mathcal{O}_S$. Then, for any $\mathcal{F} \in D^b(X \times S)$ we can tensor the filtration of the relative diagonal by $p_2^* \mathcal{F}$ to get a filtration of $p_2^* \mathcal{F} \otimes \mathcal{O}_{\Delta_X \times S}$.

Next, push the filtration for $p_2^* \mathcal{F} \otimes \mathcal{O}_{\Delta_X \times S}$ forward via p_1 . By the first paragraph of this proof, the result is a filtration for \mathcal{F} . Finally, the spectral sequence (6) is induced by using 1.11 for the filtration of \mathcal{F} which we have just obtained.

The functoriality in \mathcal{F} follows from the fact that the Postnikov tower (4) associated with a semiorthogonal decomposition is functorial in \mathcal{F} by 1.5 and from the fact that the spectral sequence associated with a Postnikov tower depends functorially on the Postnikov tower by 1.11. \square

1.3 Flatness and families of sheaves

We will often need to check flatness of families of sheaves. Most of the times such sheaves will not be coherent over the base of the family, so that it is useful to state the following classical local criterion for flatness [Eis95, thm. 6.8], where M is not assumed to be finitely generated over R .

Theorem 1.19 ([Eis95, thm 6.8]). *Suppose (R, \mathfrak{m}) is a local noetherian ring and let (S, \mathfrak{n}) be a local noetherian R -algebra such that $\mathfrak{m}S \subset \mathfrak{n}$. If M is a finitely generated S -module, then M is flat over R if and only if $\mathrm{Tor}_1^R(R/\mathfrak{m}, M) = 0$.*

We will use the following generalization of criterion 1.19. Let

$$f : X \rightarrow Y$$

be a finite type flat map of noetherian schemes. Let

$$\begin{array}{ccc} X_y & \xrightarrow{j} & X \\ \downarrow & \square & \downarrow f \\ y & \longrightarrow & Y \end{array}$$

be a cartesian diagram where y is a closed point in Y .

Corollary 1.20. *Let $\mathcal{F} \in D_{\mathrm{coh}}^-(X)$, fix $m \in \mathbb{Z}$ and assume that for any $y \rightarrow Y$ the pullback $Lj^* \mathcal{F}$ is a sheaf shifted by m . Then \mathcal{F} is the shift by m of a coherent sheaf on X which is flat over Y .*

Proof. The statement is invariant under shift, so we assume that $m = 0$. It is local on the stalk both in X and Y because taking the cohomology of a complex of sheaves and flatness are so.

We want to check that $H^i(\mathcal{F}) = 0$ for $i \neq 0$: this can be done locally around closed points of X and Y . We also want to check that $H^0(\mathcal{F})$ is flat over Y : by flatness of f , this can be checked locally around closed points of X . As f is of finite type, it sends

closed points of X into closed points of Y , so that one can localize at closed points of Y and the $\mathfrak{m}S \subset \mathfrak{n}$ assumption in theorem 1.19 holds.

In this way the statement reduces to a statement about (S, \mathfrak{n}) local ring of a closed point of X , (R, \mathfrak{m}) local ring of a closed point of Y and f a map between them sending the closed point of X into the closed point of Y . The complex of coherent sheaves \mathcal{F} becomes a complex of modules \mathcal{F} which are finitely generated over S .

There is a spectral sequence whose entries are $L^q j^*(H^p(\mathcal{F}))$ and converging to $L^{p+q} j^*(\mathcal{F})$. Note that $H^{>0}(\mathcal{F}) = 0$, as otherwise the top nonvanishing cohomology cannot be killed by any of the next differentials and would contribute non-trivially to $L^{>0} j^*(\mathcal{F})$, which we assumed is zero. As a consequence, the spectral sequence has a page

$$\begin{array}{ccccc}
\cdots & H^{-1}(\mathcal{F}) \otimes_S S/\mathfrak{m}S & H^0(\mathcal{F}) \otimes_S S/\mathfrak{m}S & 0 & \\
& \nearrow & & & \\
\cdots & \mathrm{Tor}_1^S(H^{-1}(\mathcal{F}), S/\mathfrak{m}S) & \mathrm{Tor}_1^S(H^0(\mathcal{F}), S/\mathfrak{m}S) & 0 & (7) \\
& \nearrow & & & \\
\cdots & \mathrm{Tor}_2^S(H^{-1}(\mathcal{F}), S/\mathfrak{m}S) & \mathrm{Tor}_2^S(H^0(\mathcal{F}), S/\mathfrak{m}S) & 0 &
\end{array}$$

where we have substituted $H^{>0}(\mathcal{F}) = 0$.

As we assumed that only $L^0 j^*(\mathcal{F})$ is non zero, $\mathrm{Tor}_1^S(H^0(\mathcal{F}), S/\mathfrak{m}S) = 0$. As S is flat over R , there is an isomorphism

$$\mathrm{Tor}_i^S(H^0(\mathcal{F}), S/\mathfrak{m}S) = \mathrm{Tor}_i^R(H^0(\mathcal{F}), R/\mathfrak{m}R) \quad (8)$$

so that by 1.19 for $i = 1$, $H^0(\mathcal{F})$ is flat over R .

Using (8) in the other direction, we get that $\mathrm{Tor}_i^S(H^0(\mathcal{F}), S/\mathfrak{m}S) = 0$ for all $i > 0$. This in turn implies by spectral sequence (7) that $H^{-1}(\mathcal{F}) \otimes_S S/\mathfrak{m}S = 0$. As \mathcal{F} has coherent cohomology, we can use Nakayama's lemma and obtain $H^{-1}(\mathcal{F}) = 0$.

Finally, note that if $H^i(\mathcal{F}) = 0$ for some $i \in [i_0, -1]$, by spectral sequence (7) also $H^{i_0-1}(\mathcal{F}) = 0$, so that by induction we obtain $\mathcal{F} \cong H^0(\mathcal{F})$. \square

1.4 Stability of vector bundles

Throughout section 1.4, X will be an integral locally factorial scheme whose Picard group is \mathbb{Z} . Moreover we always require that the unique ample generator $\mathcal{O}_X(1)$ has global sections. As there is a unique ample generator of $\mathrm{Pic}(X)$, we will talk about the degree of a line bundle, meaning the degree with respect to $\mathcal{O}_X(1)$.

Let us recall the definition of normalization of a vector bundle E .

Definition 1.21. The normalization E_{norm} of a vector bundle E of rank r is its unique twist such that $-r < c_1(E_{\mathrm{norm}}(k)) \leq 0$.

Equivalently, one can also say that E_{norm} is the unique twist of E with slope

$$\mu(E_{\mathrm{norm}}) = \frac{\deg c_1(E)}{\mathrm{rank}(E)}$$

in the interval $(-1, 0]$. The degree of $c_1(E)$ is taken with respect to the unique ample generator of $\text{Pic}(X)$.

The following criterion for Mumford stability goes under the name of Hoppe's criterion. It is proved in [Hop84] for vector bundles on projective spaces, but the same proof works more generally.

Theorem 1.22 (Hoppe's criterion, [Hop84]). *Assume $\mathcal{O}_X(1)$ has global sections and let E be a vector bundle of rank r on X such that for each $1 \leq s \leq r-1$ the vector bundle $(\Lambda^s E)_{\text{norm}}$ has no global sections. Then E is μ -stable.*

Proof. Assume E is not Mumford stable, i.e. that there is a sheaf F of rank $s < r$ and a sequence

$$0 \rightarrow F \rightarrow E \rightarrow Q \rightarrow 0$$

with $\mu(F) \leq \mu(E)$ and Q pure. As X is integral, E and Q are torsion free, so that F is reflexive [Har80, prop 1.1].

As X is locally factorial, we can define $\Lambda^s F$ by removing the non-locally free locus of F (which has at least codimension 2), taking the determinant on the remaining open subset of X and finally extending the line bundle back to X .

As a consequence, we have a section

$$\mathcal{O}_X \rightarrow \Lambda^s E \otimes \mathcal{O}_X(-c_1(F))$$

As $\mathcal{O}_X(1)$ has sections and X is integral, such a section induces non zero sections of $\Lambda^s E \otimes \mathcal{O}_X(-c_1(F) + k)$ for any $k \geq 0$.

Now we check that

$$(\Lambda^s E)_{\text{norm}} \cong \Lambda^s E \otimes \mathcal{O}_X(-c_1(F) + k)$$

for some $k \geq 0$. To prove it, it is enough to check that

$$\mu(\Lambda^s E \otimes \mathcal{O}_X(-c_1(F))) \leq 0$$

The left hand side of the above inequality is

$$s(\mu(E) - \mu(F)) \tag{9}$$

As F destabilizes E , (9) is not positive, so that we have constructed sections for a twist of E with non-positive slope. As $\mathcal{O}_X(1)$ has sections, also the normalized s -th exterior power of E has sections. \square

Again, let X be an integral locally factorial scheme whose Picard is generated by an ample line bundle $\mathcal{O}_X(1)$. Under these assumptions, the following lemma shows the equivalence, in the case of rank 2 bundles, of Gieseker–Maruyama stability and Mumford–Takemoto stability.

Lemma 1.23. *A vector bundle E of rank 2 on X is μ -stable if and only if it is Gieseker stable.*

Proof. Assuming that E is Gieseker stable, we prove it is μ -stable. It is enough to prove the statement for normalized rank 2 bundles, so we replace E by E_{norm} . As E has rank 2, if it is not μ -stable there is a reflexive rank 1 sheaf which destabilizes it. Assume E has a section

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow Q \rightarrow 0$$

If the section vanishes on a codimension 1 subset, as the scheme is locally factorial it vanishes on a Cartier divisor. As $\mathcal{O}_X(1)$ generates the Picard group, we can factor the section as

$$0 \rightarrow \mathcal{O}_X(n) \rightarrow E$$

for some positive n , which is impossible by μ -semistability of E .

It follows that we have an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow Q^{**} \rightarrow F \rightarrow 0$$

with F supported in codimension at least 2, showing that $c_1(Q) = 0$. Finally, the reduced Hilbert polynomial of E satisfies

$$p_E(n) = p_{\mathcal{O}_X}(n) - P_F(n)/2$$

which contradicts the Gieseker stability of E .

The other direction is standard. □

Remark 1.24. As explained in [Har80, sec.3], we have actually proved that the only strictly μ -semistable rank 2 vector bundles with $\mu = 0$ are extensions of \mathcal{O}_X by itself. As $H^1(\mathcal{O}_X)$ is the tangent space to $\text{Pic}(X)$ at the identity and as we assumed that $\text{Pic}(X) = \mathbb{Z}$, the only extension of \mathcal{O}_X by itself is $\mathcal{O}_X \oplus \mathcal{O}_X$, which is therefore the only strictly μ -semistable rank 2 vector bundle with $\mu = 0$ on X .

1.5 The Grauert–Mülich theorem

When a scheme X contains enough rational curves, one can try to recover a stable vector bundle E on X by looking at its behaviour on such curves. As all vector bundles on \mathbb{P}^1 split into a direct sum of line bundles, there is the following standard definition (e.g. [OSS80]).

Definition 1.25. The splitting type of a vector bundle E on a rational curve $L \cong \mathbb{P}^1$ is the unique sequence of integers $a_1 \leq \dots \leq a_r$ such that

$$E|_L \cong \mathcal{O}_L(a_1) \oplus \dots \oplus \mathcal{O}_L(a_r)$$

In this case, we will also write

$$\text{ST}_L(E) \cong (a_1, \dots, a_r)$$

The classical Grauert–Mülich theorem [GM75] says that for a stable vector bundle on \mathbb{P}^n the generic splitting type has no gaps. The proof mainly relies on the irreducibility of the family of lines through a point.

Among the generalizations of the Grauert–Mülich theorem, we are going to state and use the one proved by Hirschowitz in [Hir81]. Let X be a smooth projective variety of dimension n with a fixed ample line bundle $\mathcal{O}_X(1)$. Let $Z \subset X \times S$ be a non-empty proper flat family of subvarieties of X of dimension m .

$$\begin{array}{ccccc} & & Z & & \\ & p \swarrow & \downarrow & \searrow q & \\ X & \longleftarrow & X \times S & \longrightarrow & S \end{array}$$

Denote by $\mathcal{T}_{Z/X}$ the relative tangent bundle of Z over X . Denote by $[Z_s]$ the class of the general fiber of q in $H^{2n-2m}(X, \mathbb{Q})$, i.e. $p(q^{-1}(s))$ for a general point s in S .

Theorem 1.26 ([Hir81]). *Assume the following conditions hold:*

1. *p is flat out of some set of codimension at least 2 in Z .*
2. *the generic fiber of p is irreducible.*
3. *the classes $c_1(\mathcal{O}_X(1))^{n-1}$ and $[Z_s].c_1(\mathcal{O}_X(1))^{m-1}$ are proportional in $H^{2n-2}(X, \mathbb{Q})$.*

*Let $E \neq 0$ be a torsion free sheaf on X and $(\mu_1 \geq \dots \geq \mu_k)$ the slopes of the Harder–Narasimhan filtration of p^*E restricted to the general fiber of q .*

If E is semistable, then

$$\mu_i - \mu_{i+1} \leq -\mu_{\min}(\mathcal{T}_{Z/X}) \quad \text{for } i = 1, \dots, k-1$$

where $\mu_{\min}(\mathcal{T}_{Z/X})$ is the minimal slope in the Harder–Narasimhan filtration of $\mathcal{T}_{Z/X}$ relative to S .

Note that in theorem 1.26 the role of the splitting type is played by the slopes in the Harder–Narasimhan filtration of E restricted to subvarieties of X .

1.6 Koszul complexes

In this section X is a Cohen–Macaulay scheme. Given a section of a vector bundle \mathcal{E}^* , the same latter will be used to denote it and to denote the dual map from \mathcal{E} to \mathcal{O}_X .

Let s be a section of a vector bundle \mathcal{E}^* on a scheme X . There is a complex

$$\det(\mathcal{E}) \rightarrow \dots \rightarrow \Lambda^2 \mathcal{E} \rightarrow \mathcal{E} \xrightarrow{s} \mathcal{O}_X \quad (10)$$

where all maps are contractions between exterior powers induced by the section s . Complex (10) is called the Koszul complex and is denoted by $\text{Kosz}(\mathcal{E}, s)$.

Remark 1.27. The Koszul complex $\text{Kosz}(\mathcal{E}, s)$ is a DG-algebra. It follows that also its cohomology has the structure of a DG-algebra.

When X is Cohen-Macaulay it makes sense to talk about codimension, as all the local rings of X and all associated points of X have the same dimension. In this case, if the zero-locus Z of s has codimension equal to the rank r of \mathcal{E} , there is an exact sequence

$$0 \rightarrow \det(\mathcal{E}) \rightarrow \cdots \rightarrow \Lambda^2 \mathcal{E} \rightarrow \mathcal{E} \xrightarrow{s} \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$

showing that $\mathrm{Kosz}(\mathcal{E}, s)$ is quasi isomorphic to \mathcal{O}_Z .

In the case the codimension of Z is not the expected one, $\mathrm{Kosz}(\mathcal{E}, s)$ is not anymore quasi isomorphic to a sheaf. It is nonetheless possible sometimes to describe its cohomology in terms of exterior powers of an excess bundle related to the conormal bundle of Z in X . Proposition 1.28 shows how to do it.

Assume that Z is a local complete intersection of codimension c inside X , i.e. that $\dim(X) - \dim(Z) = c$ and that locally around any point of Z the ideal \mathcal{I}_Z of Z in X is generated by c elements. Then, as X is Cohen-Macaulay, a minimal set of local generators for \mathcal{I}_Z is a regular sequence, so that the conormal sheaf $\mathcal{I}_Z/\mathcal{I}_Z^2$ is locally free (see for example [Ful98, A.6.1, A.7.1]). We will denote it by $\mathcal{N}_{Z/X}^*$.

Proposition 1.28. *Let X be a Cohen-Macaulay scheme and Z a local complete intersection of codimension c in X .*

There is a vector bundle \mathcal{L}_Z on Z sitting in an exact sequence

$$0 \rightarrow \mathcal{L}_Z \rightarrow \mathcal{E}_Z \rightarrow \mathcal{N}_{Z/X}^* \rightarrow 0$$

such that the cohomology of the Koszul complex $\mathrm{Kosz}(\mathcal{E}, s)$ is isomorphic as a DG-algebra to the exterior algebra $\Lambda^\bullet \mathcal{L}_Z$.

Proof. As the schematic zero-locus of s is Z , the map s factors as

$$\mathcal{E} \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X$$

Restricting to Z surjectivity is preserved and \mathcal{I}_Z becomes the conormal bundle, so that we find an exact sequence

$$0 \rightarrow \mathcal{L}_Z \rightarrow \mathcal{E}_Z \rightarrow \mathcal{N}_{Z/X}^* \rightarrow 0 \tag{11}$$

defining a sheaf \mathcal{L}_Z . As Z is l.c.i. inside X , $\mathcal{N}_{Z/X}^*$ is a vector bundle on Z , so that \mathcal{L}_Z is a vector bundle as well.

Trivialize \mathcal{E} on an open affine subset $U \subset X$. Possibly after restricting to a smaller open subset (still denoted by U), we can split sequence (11) and lift the splitting to a commutative diagram

$$\begin{array}{ccccc} \mathcal{L}_{Z \cap U} & \longrightarrow & \mathcal{E}_{Z \cap U} & \rightleftarrows & \mathcal{N}_{Z \cap U/U}^* \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{\mathcal{L}}_{Z \cap U} & \longrightarrow & \mathcal{E}_U & \rightleftarrows & \tilde{\mathcal{N}}_{Z \cap U/U}^* \end{array} \tag{12}$$

where both $\tilde{\mathcal{L}}_{Z \cap U}$ and $\tilde{\mathcal{N}}_{Z \cap U/U}^*$ are trivial bundles on U . This is possible as the obstruction to all the liftings we want to find depends only on $H^1(\mathcal{I}_{Z \cap U/U})$, which vanishes as U is affine.

Denote the components of s with respect to the splitting (12) by s_L and s_N . In this notation the splitting (12) induces an isomorphism of DG-algebras

$$\mathrm{Kosz}(\mathcal{E}_U, s) \cong \mathrm{Kosz}(\tilde{\mathcal{L}}_Z, s_L) \otimes \mathrm{Kosz}(\tilde{\mathcal{N}}_{Z/X}^*, s_N)$$

As X is Cohen-Macaulay and s_N is locally around any point of Z a system of parameters for Z , the Koszul complex of s_N is quasi isomorphic to $\mathcal{O}_{Z \cap U}$. As s and s_N restrict to 0 on $Z \cap U$, so does s_L . It follows that there is a quasi-isomorphism of DG-algebras

$$\mathrm{Kosz}(\mathcal{E}_U, s) \cong \mathrm{Kosz}(\mathcal{L}_{Z \cap U}, 0) \cong \Lambda^\bullet \mathcal{L}_{Z \cap U} \quad (13)$$

Now we prove the global statement. We will set

$$\mathcal{H}_i := H^{-i}(\mathrm{Kosz}(\mathcal{E}, s))$$

First, we prove that $\mathcal{H}_1 \cong \mathcal{L}_Z$ by constructing a morphism and checking locally that it is an isomorphism.

Denote by K the kernel of s , i.e.

$$0 \rightarrow K \rightarrow \mathcal{E} \xrightarrow{s} \mathcal{I}_Z \rightarrow 0$$

is exact. The canonical projection from K to $\mathcal{E}|_Z$ factors via \mathcal{L}_Z . Moreover, by definition of K and of \mathcal{L}_Z , the map from K to \mathcal{L}_Z is surjective. By the exact sequence

$$\Lambda^2 \mathcal{E}|_Z \rightarrow K|_Z \rightarrow \mathcal{H}_1|_Z \rightarrow 0$$

and as the differentials of $\mathrm{Kosz}(\mathcal{E}, s)$ restrict to 0 on Z , the surjective map from $K|_Z$ to \mathcal{L}_Z factors surjectively via $\mathcal{H}_1|_Z$. We have checked locally that the cohomology of $\mathrm{Kosz}(\mathcal{E}, s)$ is supported on Z , so that $\mathcal{H}_1|_Z \cong \mathcal{H}_1$. We have therefore constructed a surjective map from \mathcal{H}_1 to \mathcal{L}_Z . Note that we know that locally both \mathcal{H}_1 and \mathcal{L}_Z are the pushforward of vector bundles on Z of rank equal to $\mathrm{rank}(\mathcal{E}) - \mathrm{codim}(Z)$. It follows the map

$$\mathcal{H}_1 \rightarrow \mathcal{L}_Z$$

is an isomorphism as it is surjective.

Finally, we prove the isomorphism of the rest of the cohomology. The diagram

$$\begin{array}{ccc} \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_1 & \longrightarrow & \mathcal{H}_k \\ \uparrow \cong & & \uparrow \vdots \\ \mathcal{L}_Z \otimes \dots \otimes \mathcal{L}_Z & \longrightarrow & \Lambda^k \mathcal{L}_Z \end{array} \quad (14)$$

can be completed to a commutative square uniquely by the universal property of exterior powers, as the multiplication of the cohomology of a DG-algebra is graded commutative. When we restrict to an arbitrary affine subscheme U , the diagram (14) still enjoys the property that there is a unique arrow from $\Lambda^k \mathcal{L}_Z|_U$ to $\mathcal{H}_k|_U$ making it commutative.

The quasi-isomorphism (13) is a map of DG-algebras, so that the induced map in cohomology is compatible with multiplication. As a consequence it coincides with the restriction of the dashed arrow of (14) to U , finally proving that

$$\Lambda^\bullet \mathcal{L}_Z \cong \mathcal{H}_\bullet$$

as DG-algebras on the whole X . □

1.7 Base change

Given a commutative square σ

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & \sigma & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array} \quad (15)$$

of schemes, there is an induced natural transformation

$$\mathrm{BC}_\sigma : Lu^* Rf_* \rightarrow Rg_* Lv^* \quad (16)$$

called the base change map. It can be defined as the composition

$$Lu^* Rf_* \xrightarrow{Lu^* Rf_*(\eta_v)} Lu^* Rf_* Rv_* Lv^* \xrightarrow{\cong} Lu^* Ru_* Rg_* Lv^* \xrightarrow{\epsilon_u(Rg_* Lv^*)} Rg_* Lv^*$$

where η_v is the counit of the adjunction for Rv_* , Lv^* , while ϵ_u is the unit for the adjunction Ru_* , Lu^* .

By interchanging the roles of f and u (and of g and v) we get another natural transformation

$$\mathrm{BC}'_\sigma : Lf^* Ru_* \rightarrow Rv_* Lg^*$$

defined analogously.

We are interested in diagrams σ such that BC_σ is an isomorphism. Theorem 1.31 describes such cartesian diagrams. In order to state it, we will need the following definition.

Definition 1.29. A cartesian diagram σ is Tor-independent if for any pair of points y', x such that $u(y') = f(x) = y$ one has

$$\mathrm{Tor}_i^{\mathcal{O}_{y,Y}}(\mathcal{O}_{x,X}, \mathcal{O}_{y',Y'}) = 0$$

for all $i > 0$.

Remark 1.30. Any cartesian diagram where u or f is flat is Tor-independent.

Theorem 1.31 ([Lip09, thm 3.10.3]). *Assume a cartesian diagram σ is given. Then the following properties are equivalent.*

1. *The natural transformation BC_σ is a functorial isomorphism.*
2. *The natural transformation BC'_σ is a functorial isomorphism.*
3. *The diagram σ is Tor-independent.*

Remark 1.32. Note that given two adjacent cartesian squares σ and τ

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y'' & \longrightarrow & Y' & \longrightarrow & Y \end{array} \quad \begin{array}{c} \sigma \quad \tau \end{array}$$

satisfying any of the equivalent conditions of theorem 1.31, it is easy to see that the outer square also satisfies them.

Next, we provide a class of Tor-independent squares.

Lemma 1.33. *Assume X, Y, Y' are smooth and irreducible and that the fiber product X' has expected dimension. Then diagram (15) is Tor-independent.*

Proof. First, we will reduce to the case where f and u are closed embeddings. Consider the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{(g,v)} & Y' \times X & \xrightarrow{p_X} & X \\ g \downarrow & & \downarrow Y' \times f & & \downarrow f \\ Y' & \xrightarrow{\Gamma_u} & Y' \times Y & \xrightarrow{p_Y} & Y \end{array} \quad (17)$$

where Γ_u is the graph of u . By remark 1.30 and as p_Y is smooth, the square on the right is Tor-independent. It follows by remark 1.32 that in order to show that σ is Tor-independent it is enough to show that the left square in (17) is so.

Note that

$$\begin{array}{ccc} X' & \xrightarrow{(g,v)} & Y' \times X \\ g \downarrow & & \downarrow Y' \times f \\ Y' & \xrightarrow{\Gamma_u} & Y' \times Y \end{array}$$

has expected dimension if and only if

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & \sigma & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

has expected dimension.

Repeating the construction with the graph of f we reduce to the case of f, u closed embeddings of smooth irreducible varieties with intersection of expected dimension. We now proceed to the computation of

$$\mathrm{Tor}_i^{\mathcal{O}_{y,Y}}(\mathcal{O}_{x,X}, \mathcal{O}_{y',Y'}) \quad (18)$$

which is clearly local on the stalk of X, Y, Y' .

As Y' is smooth and as X is integral, the structure sheaf of \mathcal{O}_X has a Koszul resolution $\text{Kosz}_{X/Y}$ by free \mathcal{O}_Y -modules which can be used to compute (18). The tensor product

$$\text{Kosz}_{X/Y} \otimes \mathcal{O}_{Y'}$$

is itself a Koszul complex with Y' , which we denote by $\text{Kosz}_{X'/Y'}$ as by definition of fiber product it has 0-th cohomology isomorphic to $\mathcal{O}_{X'}$.

Restrict around a point x' of X' . As the fiber product X' has expected dimension

$$\dim(Y) - \dim(X) = \dim(Y') - \dim(X')$$

we have that the $\dim(Y) - \dim(X)$ generators of the Koszul complex are a system of parameters for X' in Y' , which is smooth. It follows that $\text{Kosz}_{X'/Y'}$ has non-trivial cohomology only in degree 0, that is to say that all higher Tor groups (18) vanish. \square

1.8 Families of subschemes of codimension at least 2

As we will work with families of curves inside threefolds, we will need lemma 1.34. Note that S is not assumed to be reduced.

Let X be smooth and projective of dimension d_X . Let Z, W be families of subschemes of X

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & S \times X \\ & \searrow \mu_S & \downarrow \pi_S \\ & & S \end{array} \quad \begin{array}{ccc} W & \xrightarrow{\psi} & S \times X \\ & \searrow \nu_S & \downarrow \pi_S \\ & & S \end{array}$$

of dimension d_Z, d_W and flat over S .

Lemma 1.34. *Let Z, W be flat families of subschemes of X . If the ideal sheaves \mathcal{I}, \mathcal{J} of Z, W in $S \times X$ are isomorphic to each other as $\mathcal{O}_{S \times X}$ -modules, then Z and W are isomorphic families.*

Proof. We begin by proving that given a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I} & \hookrightarrow & \mathcal{O}_{S \times X} & \longrightarrow & \mathcal{O}_Z \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow t & & \downarrow \bar{t} \\ 0 & \longrightarrow & \mathcal{J} & \hookrightarrow & \mathcal{O}_{S \times X} & \longrightarrow & \mathcal{O}_W \longrightarrow 0 \end{array} \quad (19)$$

one can choose the dashed arrows to make it into a commutative diagram. Note that for this purpose it is enough to find t .

By long exact sequence, it is enough to show that

$$\text{Ext}^i(\mathcal{O}_Z, \mathcal{O}_{S \times X}) = 0$$

for $i = 0, 1$. For this to happen, it is enough that

$$R^{\leq 1} \pi_{S*} R\mathcal{H}om_{S \times X}(\varphi_* \mathcal{O}_Z, \mathcal{O}_{S \times X}) = 0$$

As φ is a proper morphism, we can rewrite the above condition as

$$R^{\leq 1} \mu_{S*} R\mathcal{H}om_Z(\mathcal{O}_Z, \varphi^! \mathcal{O}_{S \times X}) = R^{\leq 1} \pi_{S*} \varphi_* R\mathcal{H}om_Z(\mathcal{O}_Z, \varphi^! \mathcal{O}_{S \times X}) = 0 \quad (20)$$

via Grothendieck duality.

As X is smooth, $\pi_S^! \mathcal{O}_S = \omega_X[d_X]$ is a shifted line bundle. As X is smooth and μ_S is flat, φ has finite Tor-dimension, so that $\varphi^!$ commutes with tensor product (see [Nee96, Ex. 5.2]). As a consequence, we can rewrite condition (20) as

$$R^{\leq 1-d_X} \mu_{S*} R\mathcal{H}om_Z(\varphi^* \omega_X, \mu_S^! \mathcal{O}_S) = 0$$

which by Grothendieck duality for the proper map μ_S is equivalent to

$$R^{\leq 1-d_X} \mathcal{H}om_Z(R\mu_{S*}(\varphi^* \omega_X), \mathcal{O}_S) = 0$$

The hypercohomology spectral sequence yields for dimensional reasons that this last condition is true whenever $d_X - d_Z > 1$, which is one of our hypotheses.

Now that we have lifted the isomorphism between \mathcal{I} and \mathcal{J} to a morphism t in diagram (19), we want to show that t is an isomorphism. If this is true, then it induces an isomorphism \bar{t} from \mathcal{O}_Z to \mathcal{O}_W and the lemma is proved.

By snake lemma, the kernel and the cokernel of

$$\bar{t} : \mathcal{O}_Z \rightarrow \mathcal{O}_W$$

are isomorphic respectively to $\ker(t)$ and $\operatorname{coker}(t)$. We want to check that there are no non-zero morphisms from \mathcal{O}_W to $\operatorname{coker}(t)$.

Assume on the contrary that there is a morphism

$$\mathcal{O}_W \rightarrow \operatorname{coker}(t) \quad (21)$$

which is non-zero. Then there is an associated point of $\operatorname{coker}(t)$ where it is non-zero. This contradicts the fact that all associated points of W have height at least 2, while the associated primes of $\operatorname{coker}(t)$ have height at most 1.

As $\operatorname{coker}(t) = 0$, also $\ker(t)$ has to vanish (as $\mathcal{O}_{S \times X}$ is locally free), so that t is an isomorphism. \square

1.9 Closed embeddings

Given a proper map of schemes

$$f : Y \rightarrow X$$

we will often use the following criterion in order to determine whether f is a closed embedding or not.

Lemma 1.35. *Assume f is proper of relative dimension 0 and assume that the natural $\mathcal{O}_X \rightarrow f_* \mathcal{O}_Y$ is surjective. Then f is a closed embedding.*

Proof. If f is proper of relative dimension 0, then it is affine by [GD67, prop. 4.4.2]. As f is affine and as being a closed embedding is local in Y , we reduce to the case of

$$f : \operatorname{Spec}(B) \rightarrow \operatorname{Spec}(A)$$

In this case the natural $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ becomes $A \rightarrow B$. As we assume that $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is surjective, so is $A \rightarrow B$. It follows that f is a closed embedding. \square

1.10 A classical fact about (possibly nonreduced) curves in \mathbb{P}^n

Let $Z \subset \mathbb{P}^n$ be a closed subscheme such that the Hilbert polynomial $P_Z(t)$ of \mathcal{O}_Z is $d_Z t + c$. Then Z has dimension 1. We want to say that we can find a projection

$$\pi_Z : \mathbb{P}^n \rightarrow \mathbb{P}^d$$

which is an isomorphism out of a finite number of points of Z . If we denote by X the scheme-theoretic image of Z , then this is equivalent to the fact that the adjunction map

$$\mathcal{O}_X \rightarrow \pi_*\mathcal{O}_Z$$

extends to an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \pi_{Z*}\mathcal{O}_Z \rightarrow M \rightarrow 0$$

where M is supported in dimension 0.

The following proposition is analogous to the classical fact that any reduced curve is birational to an affine plane curve, but works for possibly non-reduced curves.

Proposition 1.36. *Let $Z \subset \mathbb{P}^n$ be a closed subscheme. Assume that at each generic point of Z the tangent space has dimension less or equal than d , with $d > 1$. Then for the generic linear projection π_Z to \mathbb{P}^d there is an exact sequence*

$$0 \rightarrow \mathcal{O}_X \rightarrow \pi_{Z*}\mathcal{O}_Z \rightarrow M \rightarrow 0$$

where M is supported in dimension 0.

Remark 1.37. If the Hilbert polynomial of Z is $P_Z(t) = d_Z t + c$ with $d_Z \leq d$, then the assumptions of proposition 1.36 are satisfied.

Proof. As the statement we want to prove is up to finitely many points, we can assume that Z has no components of dimension 0, i.e. it is locally Cohen-Macaulay. We will show that if $n > d$ then we can find a projection to \mathbb{P}^{n-1} as in the statement. We will denote by Z_i the components of Z , each of degree d_i .

First note that there for the general point P in \mathbb{P}^n the projection from P satisfies the following properties.

1. For each component Z_i there is a point Q_i such that the line through P and Q_i does not intersect Z in any other point.

2. For each component Z_i there is a point Q'_i such that the line through P and Q'_i does not lie in the tangent space to Z at Q'_i .

As for property (1), note that for any point Q_i the set of lines through Q_i which intersect Z in another point forms a surface. As the components are finitely many and $n > d \geq 2$, property (1) holds for the generic point in \mathbb{P}^n .

As property (1) holds for any $Q_i \in Z_i$, we will look for Q'_i among the Q_i and we will suppress the prime in Q'_i . Note that the tangent space $T_{Q_i}Z_i$ to Z_i at a general point Q_i has dimension at most d_i . It follows that the general d_i -codimensional linear space $H_i \subset \mathbb{P}^n$ containing Q_i is transversal to $T_{Q_i}Z_i$ inside $T_{Q_i}\mathbb{P}^n$, and that through the general point $P \in \mathbb{P}^n$ there is such an H_i . Any line contained in H_i is not contained in $T_{Q_i}Z_i$ and each H_i contains lines as $d_i < n$. As a consequence, property (2) holds for the general point $P \in \mathbb{P}^n$.

Let π be the linear projection from P satisfying properties (1) and (2). It restricts to a regular map $Z \rightarrow X$ which in affine coordinates in suitable neighborhoods of each $\pi(Q_i)$ becomes

$$A_X \rightarrow A_Z \tag{22}$$

Note that (22) is injective by definition of schematic image. Moreover, possibly after restricting further the affine chart in X , the map $Z \rightarrow X$ is injective on closed points by property (1). Finally, that (22) is surjective can be checked on cotangent spaces at each closed point. By property (2) we know that the map is injective on tangent spaces at each closed point, which is equivalent. It follows that (22) is an isomorphism.

As π is affine, the map (22) is the restriction of

$$0 \rightarrow \mathcal{O}_X \rightarrow \pi_{Z*}\mathcal{O}_Z \rightarrow M \rightarrow 0$$

around general points of each component of X . It follows that M is not supported on any of the components of X , so that it is supported on points. \square

Corollary 1.38. *Let $Z \subset \mathbb{P}^n$ be a closed subscheme with $P_Z(t) = d_Z t + c$. Denote by δ the length of M and by ϵ the sum of the lengths of the embedded points of X .*

1. *There is no Z such that $d_Z = 1$ and $c < 1$. If $c = 1$, Z is a line.*
2. *There is no Z such that $d_Z = 2$ and $c < 1$. If $c = 1$, Z is a plane conic.*
3. *There is no Z such that $d_Z = 3$ and $c < 0$. If $c = 0$, Z is a plane cubic. If $c = 1$, the following are equivalent*
 - *Z contains a plane cubic.*
 - *Z has an embedded point.*
 - *$h^1(\mathcal{O}_Z) \neq 0$.*

Proof. Note that $\delta = 0$ if and only if π is an isomorphism between Z and X .

Case (1) is clear as proposition 1.36 gives $P_Z = t + 1 + \epsilon + \delta$, so that $c \geq 1$ and equality holds if and only if $\delta = \epsilon = 0$.

Case (2) is clear as proposition 1.36 gives $P_Z = 2t + 1 + \epsilon + \delta$. It follows that $c \geq 1$ and equality holds if and only if $\epsilon = \delta = 0$.

In case (3) again by 1.36 one has $P_Z = P_X + \delta$ for $X \subset \mathbb{P}^3$. If the dimension of the tangent space to X is greater than 2 at each point, then X is, up to finitely many points, a triple spatial line (i.e. cut by the square of the ideal of a line). In this case $P_Z = 3t + 1 + \epsilon + \delta$, so that $c \geq 1$ and equality holds if and only if $\epsilon = \delta = 0$, i.e. if and only if both X and Z are triple spatial lines.

If the dimension of the tangent space to X is generically less than 3, project to \mathbb{P}^2 and obtain $P_Z = 3t + \epsilon + \delta$, so that $c \geq 0$ with equality if and only if Z is a plane cubic.

Finally, assume $c = 1$. Note that by the above part of the corollary, Z contains an embedded point if and only if it contains a plane elliptic curve. If Z contains a plane elliptic curve, then $h^1(\mathcal{O}_Z) = 1$ by long exact sequence. If $h^1(\mathcal{O}_Z) \neq 0$, then $h^0(\mathcal{O}_Z) > 1$, so that there is a non-surjective non-zero section of \mathcal{O}_Z , which implies that there is a non surjective

$$\varphi : \mathcal{O}_Z \rightarrow \mathcal{O}_Z$$

with cokernel \mathcal{O}_{W_1} . The ideal sheaf of W_1 in Z is a quotient of \mathcal{O}_Z , so that it is also the structure sheaf of a closed subscheme W_2 of Z . By parts (1) and (2) and additivity of the Hilbert polynomial, the degree of W_1 cannot be 1 or 2. It follows that either W_1 or W_2 has Hilbert polynomial $3t$ and is therefore an elliptic plane curve in Z . \square

Let Y_5 be the Fano threefold introduced in the next section. As we are interested in $\text{Hilb}^{dt+1}(Y_5)$, we also give the following corollary.

Corollary 1.39. *Let $d \in \{1, 2, 3\}$, then $\text{Hilb}^{dt}(Y_5)$ is empty. Any curve $Z \subset Y_5$ representing a point in $\text{Hilb}^{dt+1}(Y_5)$ has no embedded points and satisfies $h^1(\mathcal{O}_Z) = 0$.*

Proof. As for $\text{Hilb}^{dt}(Y_5)$, the only non-trivial case is $d = 3$. By part (3) of 1.38, all subschemes of \mathbb{P}^6 with Hilbert polynomial $3t$ are plane cubics. As Y_5 is cut by quadrics, if it contains a plane cubic it contains the whole plane which it spans, against Lefschetz hyperplane theorem.

As for $\text{Hilb}^{dt+1}(Y_5)$, it follows directly from part (3) of 1.38 and from the fact that Y_5 contains no plane cubics, as we have just pointed out. \square

2 The setting

Let V be a complex vector space of dimension 5. Let $A \subset \Lambda^2 V^*$ be a 3-dimensional space of 2-forms on V , and let Y_A denote the triple linear section of $\text{Gr}(2, V)$ cut by A .

The following lemma is classical and its proof can be found for example in [Isk80, ch.II, thm 1.1], where one can find the full classification of Fano threefolds of index 2. The proof which we give here is ad hoc for triple linear sections of $\text{Gr}(2, V)$, but it is simpler.

Lemma 2.1. *Y_A is a smooth threefold if and only if all forms in A have rank 4. Moreover all smooth threefolds Y_A are in the same $GL(V)$ -orbit.*

Proof. For the first part, it is a classical fact that $\mathbb{G}r(2, V^*) \subset \mathbb{P}(\Lambda^2 V^*)$ is the projective dual of $\mathbb{G}r(2, V) \subset \mathbb{P}(\Lambda^2 V)$. A very short proof is for example in [Tev05, thm 2.1] and is the following one.

The action of $GL(V)$ on $\mathbb{P}(\Lambda^2 V)$ and $\mathbb{P}(\Lambda^2 V^*)$ has only one closed orbit, i.e. respectively $\mathbb{G}r(2, V)$ and $\mathbb{G}r(2, V^*)$. As the projective dual is closed and preserved by the action of $GL(V)$, the two orbits are projectively dual to each other.

Once we know that $\mathbb{G}r(2, V^*)$ is the projective dual of $\mathbb{G}r(2, V)$, it is enough to note that $\mathbb{G}r(2, V^*)$ is the locus of forms of rank 2 and that Y_A is a smooth threefold if and only if it has 3-dimensional tangent space at any of its points, that is to say if and only if all linear sections in A are smooth.

For the second part, note that $PSL(V)$ has dimension 24, while $\mathbb{G}r(3, \Lambda^2 V^*)$ has dimension 21, so that for a general point we expect a 3-dimensional stabilizer. Denote by $SL_A(V)$ the stabilizer of A under the action of $SL(V)$ on $\mathbb{G}r(3, \Lambda^2 V)$. As the Grassmannian is irreducible, if there are points with a 3-dimensional stabilizer, then they all belong to a unique open orbit under the action of $PSL(V)$. We will prove that, for any smooth Y_A , $SL_A(V)$ is 3-dimensional. From this fact, it follows that all smooth 3-dimensional Y_A are projectively equivalent.

By its definition $SL_A(V)$ acts on A , so that there is a map

$$SL_A(V) \rightarrow SL(A) \quad (23)$$

As all forms in A have rank 4, define a $SL_A(V)$ -equivariant map

$$A \xrightarrow{\text{Ver}_2} S^2 A \xrightarrow{S^2} S^2 \Lambda^2 V^* \xrightarrow{\wedge} \Lambda^4 V^* \cong V$$

The kernel of the projection from $S^2 A$ to V is generated by a non-degenerate $SL_A(V)$ -invariant quadratic form on A^* , as one can show as follows.

Assume the kernel contains a degenerate quadratic form $a_1 \cdot a_2 \in S^2 A$, i.e. $a_1 \wedge a_2 = 0$. Fix a basis v_1, \dots, v_5 for V^* such that $a_1 = v_1 \wedge v_2 + v_3 \wedge v_4$ and denote the span of v_1, \dots, v_4 by V_4^* . It follows that

$$a_2 \in \Lambda^2 V_4^*$$

As a consequence, the pencil of 2-forms spanned by a_1 and a_2 inside $\mathbb{P}(\Lambda^2 V_4^*)$ intersects $\mathbb{G}r(2, V_4^*)$, against the assumption that all forms in A have rank 4.

Assume now that the kernel has dimension greater than 1: then there would be a degenerate quadric in it, as the set of degenerate quadrics is a divisor in $\mathbb{P}(S^2 A)$. We have therefore proved that the kernel of the projection from $S^2 A$ to V is generated by a non-degenerate quadric.

As a consequence, map (23) factors as

$$SL_A(V) \rightarrow SL_q(A) \rightarrow SL(A) \quad (24)$$

where $SL_q(A)$ is the subgroup of $SL(A)$ of transformations fixing a smooth conic, that is to say $SL(2, \mathbb{C})$.

Finally, note that, as the kernel of the $SL_A(V)$ -equivariant projection from $S^2 A$ to V is non-degenerate, there is a well defined regular map from $\mathbb{P}(A)$ to $\mathbb{P}(V)$ factoring via

$\mathbb{P}(S^2A)$. The image of this map is not contained in any hyperplane of $\mathbb{P}(V)$ as the image of Ver_2 is not contained in any hyperplane of $\mathbb{P}(S^2A)$. As the connected components of the fixed locus of an automorphism of $\mathbb{P}(V)$ are linear subspaces, an element of $PSL_A(V)$ fixing all points in the image of $\mathbb{P}(A)$ in $\mathbb{P}(V)$ has to be the identity of $\mathbb{P}(V)$. It follows that the kernel of (24) is at most discrete, so that $SL_A(V)$ is at most 3-dimensional. \square

Lemma 2.1, together with the fact that the generic triple hyperplane section of $\mathbb{G}r(2, V)$ is smooth by Bertini, justifies the following definition.

Definition 2.2. Y is the unique (up to projective equivalence) smooth threefold obtained as the intersection of $\mathbb{G}r(2, V)$ and $\mathbb{P}(A^\perp)$ inside $\mathbb{P}(\Lambda^2 V^*)$.

While proving lemma 2.1 we have introduced a map which plays an important role in the geometry of Y . We point it out in the next definition, and explain in a remark that it is an embedding.

Definition 2.3. The map σ is the composition

$$\mathbb{P}(A) \xrightarrow{\text{Ver}_2} \mathbb{P}(S^2A) \dashrightarrow \mathbb{P}(V)$$

where the first map is the Veronese embedding of degree 2 and the second one is projectivization of the composition

$$S^2A \longrightarrow S^2\Lambda^2V^* \xrightarrow{\wedge} \Lambda^4V^* \cong V$$

Remark 2.4. In 2.1 we have proved that the kernel of the projection from S^2A to V is a non-degenerate form. As the variety of bisecants to the image of Ver_2 is the variety of degenerate quadratic forms, σ is a closed embedding.

The following lemma describes the (co-)homology of Y .

Lemma 2.5. *Let M an abelian group, then*

$$H^{\text{odd}}(Y, M) = 0$$

and

$$H^0(Y, M) = H^2(Y, M) = H^4(Y, M) = H^6(Y, M) = M$$

and the same holds for homology.

Proof. The statement is true for $H^i(Y, \mathbb{Z})$ and $H_i(Y, \mathbb{Z})$ when $i \neq 3$ by Lefschetz hyperplane theorem and by Poincaré duality over \mathbb{Z} . For $i = 3$, note that by Euler characteristic H_i is torsion and by universal coefficients a non-trivial H_i would give a torsion subgroup of $H^4(Y, \mathbb{Z})$.

For arbitrary M the statement follows from universal coefficients and from the result for $M = \mathbb{Z}$. \square

As a consequence of lemma 2.5, we will often refer to classes of the Chow group of Y as to integers. For example, we will usually say that Chern classes on Y are integers. We will denote the generators of cohomology groups of Y respectively by H (the class of a hyperplane section), L (the class of a line, which is Poincaré dual to H), P (the class of a point).

On a Fano variety X , Kodaira vanishing implies

$$H^i(X, \mathcal{O}_X) = 0$$

for $i > 0$. It follows that the exponential sequence induces an isomorphism

$$c_1 : \text{Pic}(X) \rightarrow H^2(X, \mathbb{Z})$$

As a consequence, $\text{Pic}(Y)$ has a unique ample generator $\mathcal{O}_Y(1)$ induced by the unique ample generator of $\text{Gr}(2, V)$. As all ample line bundles are positive multiples of $\mathcal{O}_Y(1)$, stability of a sheaf on Y will always mean stability with respect to $\mathcal{O}_Y(1)$.

Remark 2.6. As $\mathcal{O}_Y(1)$ comes from the ambient \mathbb{P}^6 via pullback, it has global sections. Moreover, we have just shown that $\text{Pic}(Y) = \mathbb{Z}$ and Y is smooth by definition. Summing up, we have just said that Y satisfies the hypotheses of theorem 1.22 and lemma 1.23.

Vector bundles on $\text{Gr}(2, V)$ induce vector bundles on Y by restriction: we will denote the rank 2 tautological subbundle by \mathcal{U} , so that we have the following exact sequence

$$0 \rightarrow \mathcal{U} \rightarrow V \otimes \mathcal{O}_Y \rightarrow V/\mathcal{U} \rightarrow 0 \quad (25)$$

and its dual

$$0 \rightarrow \mathcal{U}^\perp \rightarrow V^* \otimes \mathcal{O}_Y \rightarrow \mathcal{U}^* \rightarrow 0 \quad (26)$$

The decomposition of the cohomology of Y is actually a shadow of a more general decomposition on Y , that of $\mathcal{D}^b(Y)$ into subcategories generated by exceptional vector bundles. More precisely, there is a semiorthogonal decomposition induced by a full exceptional collection (see [Orl91]) of the form

$$\mathcal{D}^b(Y) = \langle \mathcal{U}, \mathcal{U}^\perp, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle \quad (27)$$

We will see in proposition 2.15 that triangles (25) and (26) compute some of the mutations of the above collection, yielding other useful collections.

We will need the Chern polynomials and characters of these bundles, so we write them in the following lemma for future reference.

Lemma 2.7. *The following equalities hold in the cohomology of Y .*

$$\begin{aligned} \text{ch}(\mathcal{U}) &= 2 - H + \frac{L}{2} + \frac{P}{6} \\ \text{ch}(\mathcal{U}^\perp) &= 3 - H - \frac{L}{2} + \frac{P}{6} \\ \text{ch}(S^2 \mathcal{U}^*) &= 3 + 3H + \frac{9L}{2} - \frac{P}{6} \\ c(\mathcal{U}) &= 1 - H + 2L \\ c(\mathcal{U}^\perp) &= 1 - H + 3L - P \end{aligned} \quad (28)$$

Moreover, $H^3 = 5P$ and $\omega_Y = \mathcal{O}_Y(-2)$.

Proof. Once the Chern character of \mathcal{U} is known, all the other equalities follow from additivity of the Chern character or by multiplicativity of the Chern class with respect to exact sequences (25) and (26). On the ambient $\mathbb{G}r(2, V)$ we have that $-c_1(\mathcal{U})$ is the ample generator of the Picard group, so that the same holds by Lefschetz on Y . Moreover, on $\mathbb{G}r(2, V)$ we can easily compute $c_1(\mathcal{U})^4 c_2(\mathcal{U}) = 2P_{\mathbb{G}r}$, where $P_{\mathbb{G}r}$ is the class of a point on $\mathbb{G}r(2, V)$, so that, by functoriality of Chern classes, on Y we have $c_1(\mathcal{U})c_2(\mathcal{U}) = 2P$, which implies $c_2(\mathcal{U}) = 2L$.

The fact that $H^3 = 5P$ follows by the computation of $H_{\mathbb{G}r}^6$ on $\mathbb{G}r(2, V)$. The fact that $\omega_Y = \mathcal{O}_Y(-2)$ follows from adjunction formula and from $\omega_{\mathbb{G}r} = \mathcal{O}_{\mathbb{G}r}(-5)$. \square

The cohomology of the equivariant bundles in sequences (25) and (26) and the maps between them can be computed using Borel–Bott–Weil on the ambient $\mathbb{G}r(2, V)$ and the resolution for the structure sheaf of Y inside it.

$$0 \rightarrow \mathcal{O}_{\mathbb{G}r}(-3) \rightarrow A^* \otimes \mathcal{O}_{\mathbb{G}r}(-2) \rightarrow A \otimes \mathcal{O}_{\mathbb{G}r}(-1) \rightarrow \mathcal{O}_{\mathbb{G}r} \rightarrow \mathcal{O}_Y \rightarrow 0 \quad (29)$$

We will compute all of them in lemma 2.14.

The natural bundles on Y which we have just introduced are all μ -stable. The proof is an easy application of Hoppe’s criterion 1.22 and is carried out in 2.16.

2.1 The SL_2 -action on Y

When working on Y , one should be aware that there is an action of SL_2 on Y with an open orbit, as explained for example in [MU83]. The SL_2 -action can be induced by choosing a vector space W with $\dim W = 2$, and identifying V with $S^4 W$. Then, we get a canonical decomposition

$$\Lambda^2 V^* \cong S^2 W \oplus S^6 W$$

and we can choose $A = S^2 W$. With this choice $\mathbb{G}r(2, V) \cap \mathbb{P}(A^\perp)$ is smooth and the natural action of $SL_2(W)$ on $\Lambda^2 V$ restricts to an action on $\mathbb{G}r(2, V) \cap \mathbb{P}(A^\perp)$, as it is defined as the intersection of two $SL_2(W)$ invariant varieties.

Lemma 2.8. *Let $Y_A = \mathbb{G}r(2, V) \cap \mathbb{P}(A^\perp)$ where $A \subset \Lambda^2 V^*$ is SL_2 -equivariant. Then Y_A is smooth.*

Proof. To see that the above SL_2 -equivariant construction returns a smooth variety, we will show that all forms in A have rank 4 and then we will use lemma 2.1.

To show that all forms in A have rank 4, consider the SL_2 -equivariant projection

$$S^2 A \xrightarrow{S^2} S^2 \Lambda^2 V^* \xrightarrow{\wedge} \Lambda^4 V^* \cong V \quad (30)$$

Note that a form in A has rank 2 if and only if it is in the kernel of (30).

By decomposing it into SL_2 -irreducibles, we find

$$S^4 W \oplus \mathbb{C} \rightarrow S^4 W$$

It follows that the kernel of (30) is either a line or the whole S^2A .

If it is a line, then there is at most one degenerate form in $\mathbb{P}(A)$, which gives an irreducible non-trivial SL_2 -subrepresentation of A .

If it is the whole S^2A , then all forms in A are degenerate, so that they share a common kernel, which gives a non-trivial SL_2 -subrepresentation of V .

As V and A are by definition SL_2 -irreducible, all forms in A have rank 4, so that Y_A is smooth. \square

Remark 2.9. We have proved in 2.1 that for a smooth $\text{Gr}(2, V) \cap \mathbb{P}(A^\perp)$ the kernel of (30) is a non-degenerate conic. By lemma 2.8, this holds also for Y constructed in the SL_2 -equivariant way.

A more direct argument is the following: if the kernel of (30) is degenerate as a symmetric form on A^* , then its kernel (as a form on A^*) is SL_2 -invariant, against the fact that A is SL_2 -irreducible.

Remark 2.10. As SL_2 acts as a subgroup of $GL(V)$, sequences (25), (26) and (29) are SL_2 -equivariant.

One can describe explicitly the vector space $A \subset \Lambda^2 S^4 W$. Let us fix the notation for the SL_2 (and \mathfrak{sl}_2) action. Denote by x, y a basis for W and by $x^{n-i}y^i$ a basis for $S^n W$. Then the linear operators

$$X = x\partial_y \quad Y = y\partial_x \quad H = [X, Y] = x\partial_x - y\partial_y$$

are the standard basis for \mathfrak{sl}_2 . In this notation A is spanned by

$$3x^3y \wedge x^2y^2 - x^4 \wedge xy^3, \quad 2x^3y \wedge xy^3 - x^4 \wedge y^4, \quad 3x^2y^2 \wedge xy^3 - x^3y \wedge y^4$$

and A^\perp is spanned by

$$\begin{aligned} x^4 \wedge x^3y, \quad x^4 \wedge x^2y^2, \quad 2x^3y \wedge x^2y^2 + x^4 \wedge xy^3, \quad 8x^3y \wedge xy^3 + x^4 \wedge y^4, \\ 2x^2y^2 \wedge xy^3 + x^3y \wedge y^4, \quad x^2y^2 \wedge y^4, \quad xy^3 \wedge y^4 \end{aligned}$$

The form

$$x^4 \wedge x^2y^2 + x^2y^2 \wedge y^4 \tag{31}$$

is decomposable and belongs to A^\perp . It follows that it belongs to Y . Under the identification of A^\perp with $S^6 W$, the form (31) corresponds to

$$xy(x^4 - y^4)$$

Note that $xy(x^4 - y^4)$ has 6 distinct roots and that its stabilizer under the action of SL_2 is therefore discrete (it is a discrete group of order 48 known as the binary octahedral group). The well-known proposition 2.13 is a specialization of the following lemma to the case $f = xy(x^4 - y^4)$.

Lemma 2.11 ([MU83, lemma 1.5]). *Let f be a homogeneous polynomial of degree n in two variables. If the stabilizer of $[f] \in \mathbb{P}(S^n W)$ is a finite group and if all the roots of $f = 0$ are distinct, then the closure of the SL_2 -orbit generated by $[f]$ is decomposed into the disjoint union of SL_2 -orbits*

$$(SL_2 \cdot f) \cup (SL_2 \cdot x^5 y) \cup (SL_2 \cdot x^6)$$

Remark 2.12. The 2-dimensional orbit is the span of tangent lines to the rational normal curve of degree 6. One can check it at the point x^6 and then argue by SL_2 -equivariance.

Proposition 2.13. *The action of SL_2 on Y has the following orbit decomposition:*

- *A unique open orbit, isomorphic to SL_2/Γ , where Γ is the binary octahedral group.*
- *A unique 2-dimensional orbit, whose closure is linearly equivalent to $2H$. Moreover, it is the union of the tangent lines to the 1-dimensional orbit.*
- *A unique 1-dimensional orbit, which is a rational normal curve of degree 6.*

Proof. Using lemma 2.11 and the fact that $xy(x^4 - y^4)$ has 6 distinct roots one immediately gets the decomposition into orbits, while remark 2.12 provides the description in terms of tangent lines.

That the 2-dimensional orbit is linearly equivalent to $2H$, it follows from the decomposition into SL_2 -irreducible representations of $H^0(\mathcal{O}_Y(2))$. The reason is that the SL_2 -equivariant Koszul complex (29) provides us with an exact sequence

$$0 \rightarrow A^* \rightarrow A \otimes \Lambda^2 V^* \rightarrow H^0(\mathcal{O}_Y(2)) \rightarrow 0$$

which is SL_2 isomorphic to

$$0 \rightarrow S^2 W \rightarrow (\dots) \oplus S^0 W \rightarrow H^0(\mathcal{O}_Y(2)) \rightarrow 0$$

and implies the existence of an SL_2 -invariant divisor linearly equivalent to $2H$. As there are no SL_2 -invariant divisors linearly equivalent to H , the zero locus of the SL_2 -invariant section of $\mathcal{O}_Y(2)$ is the unique 2-dimensional orbit. \square

Via Borel–Bott–Weil, Serre duality and the sequences (25), (26) and (29) one can describe thoroughly the cohomology of the bundles on Y which come from the ambient Grassmannian (see also [Orl91]).

Lemma 2.14. *There are canonical SL_2 -equivariant isomorphisms*

- $\mathcal{U}(1) = \mathcal{U}^*$
- $H^\bullet(\mathcal{U}) = H^\bullet(\mathcal{U}(-1)) = H^\bullet(\mathcal{U}^\perp) = H^\bullet(\mathcal{U}^\perp(-1)) = H^\bullet(\mathcal{O}_Y(-1)) = 0;$
- $H^\bullet(V/\mathcal{U}(-1)) = \text{Hom}(\mathcal{U}^\perp, V/\mathcal{U}(-1)) = 0;$
- $H^\bullet(V/\mathcal{U}) = V$ and $H^\bullet(\mathcal{U}^*) = V^*;$

- $\mathrm{Hom}(\mathcal{U}, \mathcal{U}^\perp) = A$;
- $H^\bullet(\mathcal{O}_Y(1)) = \Lambda^2 V^*/A$.

Moreover, under the above identifications the composition pairing

$$A \otimes V \cong \mathrm{Hom}(\mathcal{U}, \mathcal{U}^\perp) \otimes \mathrm{Hom}(\mathcal{U}^\perp, \mathcal{O}_Y) \rightarrow \mathrm{Hom}(\mathcal{U}, \mathcal{O}_Y) \cong V^*$$

is the natural evaluation of a skew form in A on a vector of V .

Proof. The isomorphism of $\mathcal{U}(1) = \mathcal{U}^*$ follows from $\mathrm{rank}(\mathcal{U}) = 2$ and $\Lambda^2 \mathcal{U} = \mathcal{O}(-1)$. The vanishing of $H^\bullet(\mathcal{U})$, $H^\bullet(\mathcal{U}(-1))$, $H^\bullet(\mathcal{U}^\perp)$, $H^\bullet(\mathcal{U}^\perp(-1))$ and $H^\bullet(\mathcal{O}_Y(-1))$ comes from Borel–Bott–Weil and from the Koszul resolution (29). The vanishing of $H^\bullet(V/\mathcal{U}(-1))$ and $\mathrm{Hom}(\mathcal{U}^\perp, V/\mathcal{U}(-1))$ follows from the above vanishings and from the tautological sequences (25) and (26).

Again by Borel–Bott–Weil there is an SL_2 -equivariant isomorphism

$$\mathrm{Hom}_Y(\mathcal{U}, \mathcal{U}^\perp) = A \otimes \mathrm{Ext}^1(\mathcal{U}(1), \mathcal{U}^\perp) = A$$

The long exact sequences of (25) and (26) induce the other two isomorphisms, together with their SL_2 -equivariance. In the same way, one checks that $H^\bullet(\mathcal{O}_Y(1)) = \Lambda^2 V^*/A$.

To compute the composition, note that there is a unique non-zero SL_2 -equivariant map from $A \otimes V$ to V^* and that the natural pairing is SL_2 -equivariant. To understand why the pairing is non-trivial, note that it can be induced by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A \otimes \mathcal{U} & \longrightarrow & A \otimes V \otimes \mathcal{O}_Y & \longrightarrow & A \otimes V/\mathcal{U} \longrightarrow 0 \\ & & \downarrow \mathrm{ev}_{\mathcal{U}} & & \downarrow a(-, -) & & \downarrow \mathrm{ev}_{V/\mathcal{U}} \\ 0 & \longrightarrow & \mathcal{U}^\perp & \longrightarrow & V^* \otimes \mathcal{O}_Y & \longrightarrow & \mathcal{U}^* \longrightarrow 0 \end{array} \quad (32)$$

where it is always possible to draw the dashed arrow as by definition of Y the pairing $a(u_1, u_2)$ vanishes at any point $[U]$ and for any $u_1, u_2 \in U$. \square

In the next proposition we use lemma 2.14 to compute some mutations of the exceptional collection (27).

Proposition 2.15. *The following equalities hold*

$$\mathbb{R}_{\mathcal{O}_Y}(\mathcal{U}) = V/\mathcal{U}, \quad \mathbb{R}_{\mathcal{O}_Y}(\mathcal{U}^\perp) = \mathcal{U}^*, \quad \mathbb{R}_{\mathcal{U}}(V/\mathcal{U}(-1)) = \mathcal{U}^\perp. \quad (33)$$

The collections

$$\langle \mathcal{U}, \mathcal{O}_Y, \mathcal{U}(1), \mathcal{O}_Y(1) \rangle, \quad \langle \mathcal{O}_Y, V/\mathcal{U}, \mathcal{U}^*, \mathcal{O}_Y(1) \rangle \quad (34)$$

are full and exceptional.

The collection

$$\langle \mathcal{O}_Y(-1), \mathcal{U}, \mathcal{U}^\perp, \mathcal{O}_Y \rangle \quad (35)$$

is full and exceptional. Its left dual collection is

$$\langle \mathcal{O}_Y, \mathcal{U}(1), \mathcal{U}^\perp(1), \mathcal{O}_Y(1) \rangle \quad (36)$$

Proof. By Borel–Bott–Weil and resolution (29), $(V/\mathcal{U})^* = \mathcal{U}^\perp$ is acyclic on Y , that is to say that $V/\mathcal{U} \in {}^\perp \mathcal{O}_Y$. As a consequence, sequence (25) is the mutation sequence of \mathcal{U} through \mathcal{O}_Y , so that $\mathbb{R}_{\mathcal{O}_Y}(\mathcal{U}) = V/\mathcal{U}$. The dual of a mutation triangle is again a mutation triangle, so that $\mathbb{R}_{\mathcal{O}_Y}(\mathcal{U}^\perp) = \mathcal{U}^*$.

As for $\mathbb{R}_{\mathcal{U}}(V/\mathcal{U}(-1)) = \mathcal{U}^\perp$, by lemma 2.14 there is an evaluation map

$$V/\mathcal{U}(-1) \rightarrow A \otimes \mathcal{U}$$

We are now going to show that the cone of the evaluation map is \mathcal{U}^\perp . Note that by Borel–Bott–Weil and Serre duality the cone lies in the right orthogonal to \mathcal{O}_Y and $\mathcal{O}_Y(1)$. Moreover, by definition of the evaluation map, it lies in the left orthogonal to \mathcal{U} , so that by fullness of the collection (27) it is in $\langle \mathcal{U}^\perp \rangle$. It follows that the cone of the evaluation map is a direct sum of shifted copies of \mathcal{U}^\perp . As there are no maps from \mathcal{U}^\perp to $V/\mathcal{U}(-1)$, the evaluation map is injective and its cokernel is \mathcal{U}^\perp , that is to say that there is an exact sequence

$$0 \rightarrow V/\mathcal{U}(-1) \rightarrow A \otimes \mathcal{U} \rightarrow \mathcal{U}^\perp \rightarrow 0 \quad (37)$$

showing that $\mathbb{R}_{\mathcal{U}}(V/\mathcal{U}(-1)) = \mathcal{U}^\perp$.

As a consequence of (33), the collections in (34) are full and exceptional because they are mutations of (27).

The fact that (35) is exceptional follows from the more general fact that every time that there is a Serre functor \mathbb{S} for a triangulated category \mathcal{T} and an admissible subcategory \mathcal{A} is given, there is an isomorphism

$$\mathbb{S}({}^\perp \mathcal{A}) = \mathcal{A}^\perp$$

In our setting,

$$\langle \mathcal{U}, \mathcal{U}^\perp, \mathcal{O}_Y \rangle^\perp = \mathbb{S}({}^\perp \langle \mathcal{U}, \mathcal{U}^\perp, \mathcal{O}_Y \rangle) = \mathbb{S}(\langle \mathcal{O}_Y(1) \rangle) = \langle \mathcal{O}_Y(-1) \rangle$$

The left dual of collection (35) is characterized by equations (5), which hold for collections (35) and (36), as one can easily check with the aid of lemma 2.14. \square

Via Hoppe’s criterion 1.22 and lemma 2.14 we can easily check that all tautological bundles we have introduced so far are μ -stable.

Lemma 2.16. *The tautological vector bundles $\mathcal{U}, \mathcal{U}^*, \mathcal{U}^\perp, V/\mathcal{U}$ are μ -stable.*

Proof. By remark 2.6 we can use Hoppe’s criterion 1.22. As $\text{rank}(\mathcal{U}) = 2$ and $c_1(\mathcal{U}) = -1$, \mathcal{U} is a normalized bundle. To prove that \mathcal{U} is stable it is enough to check that $H^0(\mathcal{U}) = 0$, which is true by lemma 2.14.

The bundle \mathcal{U}^\perp has rank 3, so that there is an isomorphism $\Lambda^2 \mathcal{U}^\perp \cong \mathcal{O}(-1) \otimes V/\mathcal{U}$. By Hoppe’s criterion, it is enough to check $H^0(\mathcal{U}^\perp) \cong H^0(V/\mathcal{U}(-1)) = 0$, which holds true again by lemma 2.14.

As for the other two bundles, it is enough to point out that $V/\mathcal{U} = (\mathcal{U}^\perp)^*$ and that the dual of a μ -stable bundle is μ -stable. \square

2.2 Lines in Y

As Y has a canonical choice of a very ample line bundle $\mathcal{O}_Y(1)$, we will call line any subscheme L with Hilbert polynomial $h_L(t) = 1 + t$. It is well known that L is an actual line linearly embedded in \mathbb{P}^n , without any embedded points. We will need analogous statements for conics and cubics in Y : they are collected in corollary 1.39.

The Fano scheme of lines in Y is canonically identified with $\mathbb{P}(A)$. Before describing the identification, we characterize the restriction to any line of the tautological bundles coming from $\mathrm{Gr}(2, V)$.

In order to do it, recall that any vector bundle on \mathbb{P}^1 splits as a direct sum of line bundles and that we have already defined the splitting type of a vector bundle E on a line L in 1.25.

Lemma 2.17. *The splitting types of $\mathcal{U}, V/\mathcal{U}, \mathcal{U}^\perp, \mathcal{U}^*$ on any line L are as follows:*

$$\begin{aligned} \mathrm{ST}_L(\mathcal{U}) &= (-1, 0) \\ \mathrm{ST}_L(V/\mathcal{U}) &= (0, 0, 1) \\ \mathrm{ST}_L(\mathcal{U}^\perp) &= (-1, 0, 0) \\ \mathrm{ST}_L(\mathcal{U}^*) &= (0, 1) \end{aligned}$$

Proof. Both \mathcal{U} and \mathcal{U}^\perp embed in a trivial bundle by sequence (25), so that their splitting type contains no $\mathcal{O}_L(n)$ with $n > 0$. As $c_1(\mathcal{U}) = c_1(\mathcal{U}^\perp) = -1$ we have the splitting types in the thesis. The other two splitting types follow by dualization. \square

Now we are going to describe the identification of the Fano scheme of lines in Y with $\mathbb{P}(\mathrm{Hom}(\mathcal{U}, \mathcal{U}^\perp))$, which by 2.14 is canonically isomorphic to $\mathbb{P}(A)$.

Lemma 2.18. *The schematic zero locus of $v \in H^0(V/\mathcal{U}) \cong V$ is either a point or a line. It is a line if and only if $v = \ker(a)$ for some $a \in A$.*

Proof. On the ambient $\mathrm{Gr}(2, V)$, consider the dual map

$$v : \mathcal{U}^\perp \rightarrow V^* \otimes \mathcal{O}_{\mathrm{Gr}} \xrightarrow{\mathrm{ev}_v} \mathcal{O}_{\mathrm{Gr}}$$

The zero locus Z of v is isomorphic to $\mathbb{P}(V/v)$, where the embedding of $\mathbb{P}(V/v)$ into $\mathrm{Gr}(2, V)$ sends $w \in V$ to $w \wedge v \in \mathrm{Gr}(2, V)$.

Denote by Z_Y the zero locus of v on Y , and note that it is the intersection of $\mathbb{P}(V/v)$ and $\mathbb{P}(A^\perp)$ inside $\mathbb{P}(\Lambda^2 V^*)$, namely

$$Z_Y = \mathbb{P}(V/v) \cap \mathbb{P}(A^\perp) = \mathbb{P}(A(v, -)^\perp/v)$$

It follows that if $v = \ker(a)$ for some $a \in A$, then $A(v, -)^\perp$ is 3-dimensional and Z_Y is a line, otherwise $A(v, -)^\perp$ is 2-dimensional and Z_Y is a point. \square

Corollary 2.19. *Any map $a : \mathcal{U} \rightarrow \mathcal{U}^\perp$ is injective and has cokernel isomorphic to the ideal of a line.*

Proof. Any non-zero map from \mathcal{U} to \mathcal{U}^\perp is injective. Indeed, the two bundles are stable by lemma 2.16 and have slope respectively $-1/2$ and $-1/3$, so that, if a non-zero map is not injective, either the kernel destabilizes \mathcal{U} or the image destabilizes \mathcal{U}^\perp .

Extend the map a to a complex

$$\{\mathcal{U} \xrightarrow{a} \mathcal{U}^\perp \xrightarrow{\ker(a)} \mathcal{O}_Y \longrightarrow \mathcal{O}_L\} \quad (38)$$

By lemma 2.18

$$\mathcal{U}^\perp \xrightarrow{\ker(a)} \mathcal{O}_Y \longrightarrow \mathcal{O}_L \longrightarrow 0$$

is exact. As $a : \mathcal{U} \rightarrow \mathcal{U}^\perp$ is injective, the complex (38) is equivalent to the shift of a sheaf. By table (28), the Chern character of (38) vanishes, so that it is an exact sequence. \square

Not only we can associate a line with any map from \mathcal{U} to \mathcal{U}^\perp , but also the converse is true. Moreover, the correspondence works in families and gives rise to the universal family of lines in Y .

Proposition 2.20. $\mathbb{P}(A)$, $\mathbb{P}(\mathrm{Hom}(\mathcal{U}, \mathcal{U}^\perp))$ and the Hilbert scheme of lines $\mathrm{Hilb}_L(Y)$ on Y are isomorphic. Moreover, the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-3) \boxtimes \mathcal{U} \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \mathcal{U}^\perp \rightarrow \mathcal{O}_{\mathbb{P}(A) \times Y} \rightarrow \mathcal{O}_{\mathcal{L}} \rightarrow 0 \quad (39)$$

is a resolution of the structure sheaf of the universal family \mathcal{L} of lines on Y .

Proof. First, we construct a family of lines on $\mathbb{P}(A) \times Y$, later on we will prove it is the universal family.

By lemma 2.14, over $\mathbb{P}(A) \times Y$ there is a universal map from \mathcal{U} to \mathcal{U}^\perp

$$\mathcal{O}_{\mathbb{P}(A)}(-3) \boxtimes \mathcal{U} \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \mathcal{U}^\perp$$

The twist by $\mathcal{O}_{\mathbb{P}(A)}(-3)$ is chosen for compatibility with future notation. The above map extends to an SL_2 -equivariant complex

$$\mathcal{O}_{\mathbb{P}(A)}(-3) \boxtimes \mathcal{U} \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \mathcal{U}^\perp \rightarrow \mathcal{O}_{\mathbb{P}(A) \times Y} \quad (40)$$

By corollary 2.19 the restriction of (40) to any point of $\mathbb{P}(A)$ is a pure object on Y , so that by criterion 1.20 the complex (40) is equivalent to a pure object flat over $\mathbb{P}(A)$.

As a consequence we have an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-3) \boxtimes \mathcal{U} \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \mathcal{U}^\perp \rightarrow \mathcal{O}_{\mathbb{P}(A) \times Y} \rightarrow \mathcal{O}_{\mathcal{L}} \rightarrow 0 \quad (41)$$

where \mathcal{L} is a closed subscheme in $\mathbb{P}(A) \times Y$ flat over $\mathbb{P}(A)$. By (38) it is a family of lines.

Now, assume a flat family of lines $\pi : L \rightarrow S$ over a base S is given. We want to construct a map $f : S \rightarrow \mathbb{P}(A)$ associated to L . By theorem 1.18 there is a relative Beilinson spectral sequence

$$R^q \pi_* (\mathcal{O}_L \otimes (\vee E_{n-p})^*) \boxtimes E_{p+1} \Rightarrow \mathcal{O}_L$$

with respect to the full exceptional collection $\mathcal{O}_Y(-1), \mathcal{U}, \mathcal{U}^\perp, \mathcal{O}_Y$ and its left dual collection $\mathcal{O}_Y, \mathcal{U}^*, \mathcal{U}^\perp(1), \mathcal{O}_Y(1)$ (see proposition 2.15).

We will now discuss the vanishing of most of the entries of the relative Beilinson spectral sequence. Note that we can substitute $R^{>1}\pi_*(\mathcal{O}_L \otimes -) = 0$ as the relative dimension of L over S is 1. It follows that the first page of the spectral sequence is

$$\begin{array}{ccccccc}
R^1\pi_*\mathcal{O}_L(-1) \otimes \mathcal{O}_Y(-1) & \rightarrow & R^1\pi_*V/\mathcal{U}(-1)|_L \otimes \mathcal{U} & \rightarrow & R^1\pi_*\mathcal{U}|_L \otimes \mathcal{U}^\perp & \rightarrow & R^1\pi_*\mathcal{O}_L \otimes \mathcal{O}_Y \\
& & \searrow d_2 & & \searrow d_2 & & \\
\pi_*\mathcal{O}_L(-1) \otimes \mathcal{O}_Y(-1) & \longrightarrow & \pi_*V/\mathcal{U}(-1)|_L \otimes \mathcal{U} & \longrightarrow & \pi_*\mathcal{U}|_L \otimes \mathcal{U}^\perp & \longrightarrow & \pi_*\mathcal{O}_L \otimes \mathcal{O}_Y
\end{array} \tag{42}$$

As $\mathcal{O}_L(-1)$ is acyclic, the column $p = 0$ vanishes. By base change and lemma 2.17, we can use theorem 1.20 on $R\pi_*(\mathcal{O}_L \otimes V/\mathcal{U}(-1))$ and deduce that it is equivalent to a line bundle $\mathcal{L}_\mathcal{U}$. Analogously, $R\pi_*(\mathcal{O}_L \otimes \mathcal{U}) \cong \mathcal{L}_\mathcal{U}^\perp$ and $R\pi_*(\mathcal{O}_L) \cong \mathcal{O}_S$. The spectral sequence (42) becomes

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
0 & & \pi_*V/\mathcal{U}(-1)|_L \otimes \mathcal{U} & \longrightarrow & \pi_*\mathcal{U}|_L \otimes \mathcal{U}^\perp & \longrightarrow & \pi_*\mathcal{O}_L \otimes \mathcal{O}_Y
\end{array}$$

Altogether, we have an exact sequence

$$0 \rightarrow \mathcal{L}_\mathcal{U} \boxtimes \mathcal{U} \rightarrow \mathcal{L}_\mathcal{U}^\perp \boxtimes \mathcal{U}^\perp \rightarrow \mathcal{O}_{S \times Y} \rightarrow \mathcal{O}_L \rightarrow 0 \tag{43}$$

on $S \times Y$.

Take $\mathcal{H}om(\mathcal{U}, -)$ of the first map of sequence (43) and push it forward to S . The result is a map

$$\mathcal{L}_\mathcal{U} \rightarrow A \otimes \mathcal{L}_\mathcal{U}^\perp \tag{44}$$

As \mathcal{O}_L is flat over S , when we base change to $s \times Y$ the sequence (43) is still exact, so that the map (39) is an injection of vector bundles, and therefore defines a unique map $f : S \rightarrow \mathbb{P}(A)$ such that the pullback of the universal

$$\mathcal{O}_{\mathbb{P}(A)}(-1) \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(A)}$$

is (44) twisted by $\mathcal{L}_\mathcal{U}^*$.

So far we have constructed two correspondences: one associates a map $S \rightarrow \mathbb{P}(A)$ with a family $\pi : L \rightarrow S$, the other associates a family to a map $S \rightarrow \mathbb{P}(A)$ by pulling back \mathcal{L} . We need to check that they are natural and that they are mutually inverse.

The naturality follows easily from their definitions and from the functoriality of the Beilinson spectral sequence, which is discussed in theorem 1.18.

The fact that if we start from a map from $S \rightarrow \mathbb{P}(A)$, pullback the universal \mathcal{L} and use it to induce a map from $S \rightarrow \mathbb{P}(A)$, then we recover the initial map is also a consequence of the functoriality of the Beilinson spectral sequence.

The fact that if we start from a family of lines $\pi : L \rightarrow S$, induce a map $S \rightarrow \mathbb{P}(A)$ and pullback the universal \mathcal{L} via this map, then we recover the initial map can be explained in the following way. By definition of the two correspondences we recover the ideal sheaf of the family $L \subset S \times Y$. As $\text{codim}(L) = 2$ and by lemma 1.34, the ideal sheaf determines the subscheme L . \square

Remark 2.21. In proposition 2.20 we have established an SL_2 -equivariant isomorphism between the Hilbert scheme of lines on Y and $\mathbb{P}(A)$. Under the action of SL_2 , $\mathbb{P}(A)$ has two orbits: the open one and the conic \mathcal{Q} which is the zero locus of unique invariant form $q \in S^2 A^*$.

The lines corresponding to points on \mathcal{Q} are known in the literature as special lines, and we will see in proposition 2.27 that they are characterized by the fact that their normal bundle in Y is not trivial.

The following definition sets the notation which we are going to use for the identification of the Hilbert scheme of lines and of $\mathbb{P}(A)$.

Definition 2.22. Given an element $a \in A$, denote by L_a the corresponding line in Y . Given a line L in Y , denote by a_L any element in A such that L corresponds to $[a_L] \in \mathbb{P}(A)$.

We are now going to state and prove a few consequences of proposition 2.20. We will denote the projections from the universal line \mathcal{L} to $\mathbb{P}(A)$ and Y by r_A and by r_Y , as in the following diagram

$$\begin{array}{ccc} & \mathcal{L} & \\ r_A \swarrow & & \searrow r_Y \\ \mathbb{P}(A) & & Y \end{array}$$

In what follows, given a point $P \in Y$, U_P is the fiber of \mathcal{U} at P . All lines in Y arise by the following simple construction.

Corollary 2.23. *Given two points $P, Q \in Y$, there is a line connecting them if and only if U_P and U_Q have non-trivial intersection. In this case the line is given by*

$$\mathbb{P}\left(\frac{U_P + U_Q}{\ker a}\right) \subset \text{Gr}(2, V)$$

Proof. By proposition 2.20, for P and Q to belong to a line L it is equivalent that there exists $a \in A$ such that $\ker a = 0$ in both V/U_P and V/U_Q . Therefore, if P and Q belong to L , the intersection $U_P \cap U_Q$ is not trivial. In the other direction, if there exists $v \in U_P \cap U_Q$, then $A(v, U_P + U_Q) = 0$ inside V^* , so that $\mathbb{P}((U_P + U_Q)/v)$ is a line which embeds in Y and contains P, Q . \square

Corollary 2.24. *The map $r_Y : \mathcal{L} \rightarrow Y$ is a 3 to 1 cover. The multiplicity of preimages is $(1, 1, 1)$ over the open orbit, $(2, 1)$ over the divisorial orbit and (3) over the closed orbit.*

Proof. It is enough to prove that the family of lines through any point is finite and that the rank of $r_{A*}\mathcal{O}_{\mathcal{L}}$ is 3.

The family of lines through any point is finite by the following argument. Assume there are infinitely many lines through $P \in Y$. By proposition 2.20 all lines have a resolution of the form (38), so that each vector in U_P is the kernel of some $a \in A$. It follows that $\mathbb{P}(U_P) \subset \sigma(A)$, against the fact that the $\sigma(A)$, being a surface of degree 2, does not contain lines.

As for $r_{A*}\mathcal{O}_{\mathcal{L}}$, it is easily computed by pushing forward the resolution (39). The resulting spectral sequence shows that $r_{A*}\mathcal{O}_{\mathcal{L}} \cong \mathcal{O} \oplus \mathcal{U}$.

By proposition 2.20, the ramification divisor of r_Y is linearly equivalent to $r_A^*\mathcal{O}_{\mathbb{P}(A)}(2)$. As it is SL_2 -invariant, it has to be the preimage of special lines $r_A^{-1}(\mathcal{Q})$. For the same reason the ramification type over the 2-dimensional orbit in Y is $(2, 1)$.

Finally, lines intersecting the closed orbit span an SL_2 -invariant divisor, so that by lemma 2.11 and remark 2.12 they are all tangent to the closed orbit. \square

Remark 2.25. By corollary 2.24, the family of lines through a general point in Y is not connected, so that condition (2) of theorem 1.26 is not verified.

We will now describe the intersection pairing between lines in Y . Given two distinct lines L_1 and L_2 with ideals I_1 and I_2 there are several equivalent ways to check whether they intersect or not. We look for a condition which is minimal among the ones which cut the closure of the locus of intersecting distinct lines inside $\mathbb{P}(A) \times \mathbb{P}(A)$.

Let SL_2 act on $\mathbb{P}(A) \times \mathbb{P}(A)$ componentwise. Then the space $H^0(\mathcal{O}(1, 1))$ contains a unique SL_2 -invariant line. Indeed, by Littlewood–Richardson it is SL_2 -isomorphic to $A^* \otimes A^* = V \oplus A \oplus \mathbb{C}$. Under the decomposition $A^* \otimes A^* \cong \Lambda^2 A^* \oplus S^2 A^*$, the SL_2 -invariant element is symmetric as $\Lambda^2 A^* \cong A$.

As a consequence, we have selected an SL_2 -invariant effective divisor inside $\mathbb{P}(A) \times \mathbb{P}(A)$. We will denote it by IQ . Note that, by means of the SL_2 -invariant quadric q , we can identify A and A^* , so that we find an SL_2 -equivariantly embedded flag variety $\mathbb{F}\ell(1, 2; A)$ inside $\mathbb{P}(A) \times \mathbb{P}(A)$. Moreover the flag variety is an effective divisor of type $(1, 1)$, so that by uniqueness of such a divisor we have

$$IQ = \mathbb{F}\ell(1, 2; A) \subset \mathbb{P}(A) \times \mathbb{P}(A) \quad (45)$$

Proposition 2.26. *IQ is the closure of the locus of distinct intersecting lines.*

Proof. The SL_2 -invariant conic in $S^2 A^*$ induces an SL_2 -invariant section

$$\mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \mathcal{O}_{\mathbb{P}(A)}(-2) \rightarrow V \boxtimes V \rightarrow A^* \otimes \mathcal{O}_{\mathbb{P}(A) \times \mathbb{P}(A)} \quad (46)$$

over $\mathbb{P}(A) \times \mathbb{P}(A)$. The composition of (46) with the SL_2 -invariant surjection

$$A^* \otimes \mathcal{O}_{\mathbb{P}(A) \times \mathbb{P}(A)} \rightarrow \mathcal{O}_{\mathbb{P}(A)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(A)} \oplus \mathcal{O}_{\mathbb{P}(A)} \boxtimes \mathcal{O}_{\mathbb{P}(A)}(1) \quad (47)$$

vanishes: one can check it fiberwise at (a_1, a_2) , where the composition of (46) and (47) becomes

$$a_1^2 \otimes a_2^2 \mapsto \ker a_1 \otimes \ker a_2 \mapsto \ker a_1 \wedge \ker a_2 \mapsto a_1(\ker a_1, \ker a_2) \oplus a_2(\ker a_1, \ker a_2) = 0$$

As a consequence, map (46) factors via the kernel of (47). This provides us with an SL_2 -invariant

$$\mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \mathcal{O}_{\mathbb{P}(A)}(-2) \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-1) \boxtimes \mathcal{O}_{\mathbb{P}(A)}(-1)$$

which vanishes at (a_1, a_2) if and only if $A(\ker a_1, \ker a_2) = 0$, that is to say if and only if $\ker a_1 \wedge \ker a_2 \in Y$. By description 2.23 and for $a_1 \neq a_2$, this happens if and only if L_{a_1} and L_{a_2} intersect. \square

Proposition 2.27. *A line $L = L_a$ has normal bundle $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$ if and only if $(a, a) \in IQ$, otherwise its normal bundle is trivial.*

Proof. The derived tensor product

$$\mathcal{O}_L \overset{L}{\otimes} \mathcal{O}_L$$

is equivalent to $\mathcal{O}_L[2] \oplus \mathcal{N}_{L/Y}^*[1] \oplus \mathcal{O}_L$. By resolution (43), it is also equivalent to

$$\mathcal{O}_L(-1) \oplus \mathcal{O}_L \rightarrow \mathcal{O}_L(-1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L \rightarrow \mathcal{O}_L$$

By removing the Tor_0 and Tor_2 components, we are left with an exact sequence

$$0 \rightarrow \mathcal{O}_L(-1) \rightarrow \mathcal{O}_L(-1) \oplus \mathcal{O}_L \oplus \mathcal{O}_L \rightarrow \mathcal{N}_{L/Y}^*[1] \rightarrow 0$$

which shows that the normal bundle is nontrivial if and only if the $\mathcal{O}_L(-1)$ component of the leftmost map in the above sequence vanishes. By the description of lemma 2.14 for the maps from \mathcal{U} to \mathcal{U}^\perp , this condition is equivalent to the fact that

$$A(\kappa(a), \kappa(a)) = 0$$

Note that this condition is equivalent to the vanishing of the composition

$$\mathcal{O}_{\mathbb{P}(A)}(-1) \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(A)} \rightarrow \Lambda^2 V^* \otimes \mathcal{O}_{\mathbb{P}(A)} \rightarrow \Lambda^2 \kappa^* \mathcal{K}^* \quad (48)$$

Moreover, as $A(\sigma(a), \kappa(a)) = 0$ and $\sigma(a) \in \kappa(a)$, the composition (48) factors via

$$\mathcal{O}_{\mathbb{P}(A)}(1) \cong \Lambda^2(\kappa^* \mathcal{K} / \sigma^* \mathcal{O}_{\mathbb{P}(V)}(-1))^* \rightarrow \Lambda^2 \kappa^* \mathcal{K}^*$$

where the isomorphism on the left follows from lemma 3.1. As a consequence, the set of lines with nontrivial normal bundle correspond to the zero-locus of a section of $\mathcal{O}_{\mathbb{P}(A)}(2)$. As the unique SL_2 -invariant quadric is IQ , it follows that the normal bundle to L_a is non-trivial if and only if (a, a) is in IQ . \square

Remark 2.28. In other words, two distinct lines intersect if and only if they are polar with respect to the unique SL_2 -invariant conic $\mathcal{Q} \subset \mathbb{P}(A)$. Proposition 2.27 says that (a, a) lies on IQ if and only if there is an infinitesimal deformation of L_a fixing a point of L_a .

The resolution (43) for the structure sheaf of the universal line allows us to compute the action of the Fourier–Mukai functor

$$\Phi_{\mathcal{L}} : D^b(Y) \rightarrow D^b(\mathbb{P}(A))$$

with kernel the structure sheaf of \mathcal{L} . We compute the transforms of a few bundles which we will need later.

Lemma 2.29. *The following equalities hold*

$$\Phi_{\mathcal{L}}(\mathcal{U}(-1)) = \mathcal{O}_{\mathbb{P}(A)}(-3)[1], \quad \Phi_{\mathcal{L}}(\mathcal{U}^\perp(-1)) = \mathcal{O}_{\mathbb{P}(A)}(-2)[1] \quad (49)$$

Moreover,

$$\Phi_{\mathcal{L}} : A \cong \mathrm{Hom}(\mathcal{U}(-1), \mathcal{U}^\perp(-1)) \rightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}(A)}(-3), \mathcal{O}_{\mathbb{P}(A)}(-2)) \cong A^* \quad (50)$$

is the unique (up to scalar) SL_2 -equivariant isomorphism.

Proof. Both equalities in (49) are proved by means of the Grothendieck spectral sequence and using the resolution (43). The entries in the spectral sequence can be computed via Künneth formula and Borel–Bott–Weil formula and in both cases there is only one non-vanishing entry.

As for the isomorphism (50), apply $\Phi_{\mathcal{L}}$ on the mutation triangle (37)

$$0 \rightarrow V/\mathcal{U}(-1) \rightarrow A \otimes \mathcal{U} \xrightarrow{\mathrm{ev}_{\mathcal{U}}} \mathcal{U}^\perp \rightarrow 0$$

By (49) we find a distinguished triangle

$$\Phi_{\mathcal{L}}(V/\mathcal{U}(-1))[-1] \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(A)}(-3) \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-2)$$

If (50) is zero, then $\Phi_{\mathcal{L}}(V/\mathcal{U}(-1))$ is the direct sum of two line bundles with different shifts. As we know by lemma 2.17 and by base change that $\Phi_{\mathcal{L}}(V/\mathcal{U}(-1))$ restricts to any point $a \in \mathbb{P}(A)$ as $\mathbb{C} \oplus \mathbb{C}$, this is not possible, so that (50) has to be an isomorphism. \square

Remark 2.30. Actually, we have also proved that $\Phi_{\mathcal{L}}(V/\mathcal{U}(-1))$ is $\Omega_{\mathbb{P}(A)}(-2)[1]$.

2.3 Conics in Y

We will prove in proposition 2.32 that the Hilbert scheme of conics in Y is canonically identified with $\mathbb{P}(V^*)$. By conic on Y we mean any subscheme $C \subset Y$ such that $\chi(\mathcal{O}_C(n)) = 2t + 1$.

Remark 2.31. Note that the only such subschemes in projective spaces are plane conics (corollary 1.38 part (2)). It follows that they satisfy $h^1(\mathcal{O}_C) = 0$ and they have no embedded points.

When the conic C is smooth, it makes sense to talk about splitting type of a vector bundle on it. Still, when C is singular the splitting type on components (or on the line supporting the non-reduced conic) does not characterize uniquely the vector bundle. In these cases it is better to have cohomological properties, which in the smooth case allow to recover the splitting type. We pack such properties in the decomposition of \mathcal{O}_C with respect to the full exceptional collection (35).

With some help from lemmas 2.36 – 2.39, we prove proposition 2.32, which describes the universal conic $\mathcal{C} \subset \mathbb{P}(V^*) \times Y$ as the zero locus of a regular section of $\mathcal{O}_{\mathbb{P}(V^*)}(1) \boxtimes \mathcal{U}^*$, as $\mathbb{P}_Y(\mathcal{U}^\perp)$ and as the restriction of $\mathbb{F}l(2, 4; V) \subset \mathbb{P}(V^*) \times \mathbb{G}r(2, V)$ to $\mathbb{P}(V^*) \times Y$.

Proposition 2.32. *The Hilbert scheme of conics Hilb_Y^{2t+1} is canonically isomorphic to $\mathbb{P}(V^*) \cong \mathbb{P}(H^0(\mathcal{U}^*))$. The universal conic \mathcal{C} has a resolution*

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(V^*)}(-2) \boxtimes \mathcal{O}_Y(-1) \rightarrow \mathcal{O}_{\mathbb{P}(V^*)}(-1) \boxtimes \mathcal{U} \rightarrow \mathcal{O}_{\mathbb{P}(V^*) \times Y} \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow 0 \quad (51)$$

on $\mathbb{P}(V^*) \times Y$. The schemes

$$\mathcal{C} \cong \mathbb{P}(\mathcal{U}^\perp) \cong \mathbb{F}l(2, 4, V)|_{\mathbb{P}(V^*) \times Y} \subset \mathbb{F}l(2, 4, V) \subset \mathbb{P}(V^*) \times \mathbb{G}r(2, V) \quad (52)$$

are isomorphic as subschemes of $\mathbb{P}(V^*) \times Y$.

Proof. Given a flat family $C \rightarrow S \times Y$ of conics in Y , take the relative Beilinson spectral sequence 1.18 with respect to the full exceptional collection (35), i.e. $\mathcal{O}(-1), \mathcal{U}, \mathcal{U}^\perp, \mathcal{O}$.

Denote by π_S the projection from $C \rightarrow S$ and by π_Y the projection $C \rightarrow Y$. As the relative dimension of π_S is 1, when computing the entries of the relative Beilinson spectral sequence for \mathcal{O}_C we will restrict to $R^0\pi_{S*}$ and $R^1\pi_{S*}$.

We start from $R\pi_{S*}(\mathcal{O}_C)$: by remark 2.31, $H^1(\mathcal{O}_{C_s}) = 0$ for any $s \in S$, so that by base change $R^1\pi_{S*}(\mathcal{O}_C) = 0$. As a consequence, $R^0\pi_{S*}(\mathcal{O}_C)$ is locally free of rank 1. As it has a never vanishing section induced by the 1 section of \mathcal{O}_C , it is \mathcal{O}_S .

The entries in the second column are the cohomology of $R\pi_{S*}(\mathcal{O}_C \otimes \pi_Y^* \mathcal{U})$. By lemma 2.38, for any conic C_s the cohomology $H^\bullet(C_s, \mathcal{U})$ vanishes, so that also $R\pi_{S*}(\mathcal{O}_C \otimes \pi_Y^* \mathcal{U})$ vanishes.

The entries in the third column are the cohomology of $R\pi_{S*}(\mathcal{O}_C \otimes \pi_Y^* V/\mathcal{U}(-1))$. By lemma 2.36 and by Serre duality, for any conic C_s

$$H^\bullet(V/\mathcal{U}(-1)|_{C_s}) = H^\bullet(\mathcal{U}^\perp|_{C_s}[3])^*$$

By lemma 2.37, for any conic C_s we find $H^\bullet(C_s, V/\mathcal{U}(-1)) = \mathbb{C}[-1]$. By lemma 1.20, $R\pi_{S*}(\mathcal{O}_C \otimes \pi_Y^* V/\mathcal{U}(-1))$ is quasi isomorphic to the shift of a line bundle. Denote it by $\mathcal{L}_{\mathcal{U}}$.

The entries of the last column are the cohomology of $R\pi_{S*}(\mathcal{O}_C(-1))$. By lemma 2.36 and remark 2.31, for any conic C_s we find $H^\bullet(C_s, \mathcal{O}_Y(-1)) = \mathbb{C}[-1]$. Again, by lemma 1.20, $R\pi_{S*}(\mathcal{O}_C(-1))$ is quasi isomorphic to the shift of a line bundle. Denote it by $\mathcal{L}_{\mathcal{O}(-1)}$.

As the Beilinson spectral sequence 1.18 converges to \mathcal{O}_C , there is an exact sequence

$$0 \rightarrow \mathcal{L}_{\mathcal{O}(-1)} \boxtimes \mathcal{O}_Y(-1) \rightarrow \mathcal{L}_{\mathcal{U}} \boxtimes \mathcal{U} \rightarrow \mathcal{O}_{S \times Y} \rightarrow \mathcal{O}_C \rightarrow 0 \quad (53)$$

Twisting by $\mathcal{O}_Y(1)$ and pushing forward to S , we get an exact sequence

$$0 \rightarrow \mathcal{L}_{\mathcal{O}(-1)} \rightarrow \mathcal{L}_{\mathcal{U}} \otimes V^* \rightarrow \mathcal{O}_S \otimes \Lambda^2 V^* / A \rightarrow \pi_*(\mathcal{O}_C(1)) \rightarrow 0 \quad (54)$$

To get a map from S to $\mathbb{P}(V^*)$ we still need to prove that the first map in (54) is an injection of vector bundles. To prove it, we base change to $s \in S$ by first twisting (53) by $\mathcal{O}_Y(1)$, then restricting the resulting complex to $s \times Y$ and finally pushing forward to s . By flatness of \mathcal{O}_C over S , we find an exact sequence

$$0 \rightarrow \mathbb{C} \rightarrow V^* \rightarrow \Lambda^2 V^* / A \rightarrow H^0(\mathcal{O}_Y(1)|_C) \rightarrow 0$$

which shows that

$$\mathcal{L}_{\mathcal{O}(-1)} \rightarrow \mathcal{L}_{\mathcal{U}} \otimes V^*$$

is an injection of vector bundles. In the above way, given a family of conics C , we have constructed a map from S to $\mathbb{P}(V^*)$

We will now construct a family which will be the universal family of conics. The structure sheaf of the subscheme

$$\mathbb{P}_Y(\mathcal{U}^\perp) \subset \mathbb{P}(V^*) \times Y$$

is cut by the Koszul complex

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(V^*)}(-2) \boxtimes \mathcal{O}_Y(-1) \rightarrow \mathcal{O}_{\mathbb{P}(V^*)}(-1) \boxtimes \mathcal{U} \rightarrow \mathcal{O}_{\mathbb{P}(V^*) \times Y}$$

of the SL_2 -invariant section of $\mathcal{O}_{\mathbb{P}(V^*)}(1) \boxtimes \mathcal{U}^*$.

All fibers of the projection

$$\mathbb{P}_Y(\mathcal{U}^\perp) \rightarrow \mathbb{P}(V^*)$$

have dimension 1 as there is no map from \mathcal{U} to \mathcal{O}_Y vanishing on a divisor. It follows that $\mathbb{P}_Y(\mathcal{U}^\perp)$ is flat over $\mathbb{P}(V^*)$ and that therefore

$$0 \rightarrow \mathcal{O}_Y(-1) \rightarrow \mathcal{U} \rightarrow \mathcal{O}_Y$$

is exact for each $v \times Y$ inside $\mathbb{P}(V^*) \times Y$. By table (28), $\mathbb{P}_Y(\mathcal{U}^\perp)$ is a family of conics.

What is still left to prove is that the two constructions are functorial and that they are inverse to each other: both facts can be proved in the same way they were proved for lines in proposition 2.20, again by means of lemma 1.34.

As for the isomorphism of $\mathbb{P}_Y(\mathcal{U}^\perp)$ and the restriction of the flag variety $\mathbb{Fl}(2, 4, V)$ to $\mathbb{P}(V^*) \times Y$, this is a direct consequence of the fact that $\mathbb{Fl}(2, 4, V)$ embeds into $\mathbb{P}(V^*) \times \mathbb{Gr}(2, V)$ as $\mathbb{P}_{\mathbb{Gr}}(\mathcal{U}^\perp)$ and that restriction commutes with projectivization. \square

Note that by proving proposition 2.32 we have also proved the following corollary.

Corollary 2.33. *Any section of \mathcal{U}^* is regular and fits in a Koszul complex*

$$0 \rightarrow \mathcal{O}_Y(-1) \rightarrow \mathcal{U} \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_C \rightarrow 0 \quad (55)$$

where C is a conic in Y .

Proof. In 2.32 we have shown that given $\mathcal{U} \rightarrow \mathcal{O}_Y$ the cokernel is always a conic, so that any section of \mathcal{U}^* is regular. Then the Koszul complex is exact. \square

The following definition sets the notation which we are going to use for the identification of the Hilbert scheme of conics and of $\mathbb{P}(V^*)$.

Definition 2.34. Given an element $w \in V^*$, denote by L_w the corresponding conic in Y . Given a conic C in Y , denote by w_C any element in V^* such that C corresponds to $[w_C] \in \mathbb{P}(V^*)$.

Actually, proposition 2.32 also says that any conic is obtained by choosing a non zero vector $v \in V^*$ and taking $\text{Gr}(2, \ker v) \cap \mathbb{P}(A^\perp)$ inside $\mathbb{P}(\Lambda^2 V)$. Viceversa, every time we choose $v \in V^*$ and intersect $\text{Gr}(2, \ker v)$ embedded via Plücker with $\mathbb{P}(A^\perp)$, we find a conic in Y , just because $\text{Gr}(2, 4)$ itself is a quadric in \mathbb{P}^5 .

Corollary 2.35. *The zero locus of any section of \mathcal{U}^* is a conic.*

Proof. The last map in the resolution (51) is the universal map from \mathcal{U} to \mathcal{O}_Y . \square

Here are the lemmas we have used in the proof of 2.32. Recall that a conic is any subscheme C of Y whose Chern character is $2L - P$, or equivalently whose Hilbert polynomial is $2t + 1$.

Lemma 2.36. *The canonical bundle of any conic C is the restriction of $\mathcal{O}_Y(-1)$ to C .*

Proof. As $\mathcal{O}_Y(-1)$ is the restriction of $\mathcal{O}(-1)$ from a projective space, it is enough to prove the same statement for a conic C embedded in \mathbb{P}^n . Moreover, any $C \in \mathbb{P}^n$ factors via a \mathbb{P}^2 spanned by $H^0(\mathcal{O}_C(1))^*$ by remark 2.31.

A conic in \mathbb{P}^2 is a Cartier divisor (as it has no embedded points by remark 2.31), so that by adjunction

$$\omega_C \cong \omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(2)|_C \cong \mathcal{O}_{\mathbb{P}^2}(-1)|_C$$

which is the thesis. \square

Lemma 2.37. *The following relations hold on Y for any conic C :*

$$\begin{array}{ll} \chi(\mathcal{U}^\perp|_C) = 1 & \chi(\mathcal{U}^*|_C) = 4 \\ h^0(\mathcal{U}^\perp|_C) = 1 & h^0(\mathcal{U}^*|_C) = 4 \\ h^1(\mathcal{U}^\perp|_C) = 0 & h^1(\mathcal{U}^*|_C) = 0 \end{array}$$

Proof. To compute $\chi(\mathcal{U}^\perp|_C)$ use Riemann–Roch.

To check $h^0(\mathcal{U}^\perp|_C) = 1$ it is then enough to check $h^0(\mathcal{U}^\perp|_C) \leq 1$. Assume this is not the case, i.e. that there is $B \subset H^0(\mathcal{U}^\perp|_C)$, $\dim(B) = 2$. This implies that C is schematically contained in $Y' = \text{Gr}(2, B^\perp) \cap Y \subset \text{Gr}(2, V)$. As Y is a linear section of $\text{Gr}(2, V)$, $Y' \subset Y$ is a linear section of $\text{Gr}(2, B^\perp) = \mathbb{P}^2$ linearly embedded in $\mathbb{P}(\Lambda^2 V^*)$. Moreover, we know by Lefschetz that Y contains no \mathbb{P}^2 , so that Y' is either a line or a point. As C is a

conic and Y' is a line or a point, C can't be a subscheme of Y' , so that it must be $h^0(\mathcal{U}^\perp|_C) \leq 1$.

As for $h^1(\mathcal{U}^\perp|_C) = 0$, it follows by subtracting the other two results and using $\dim(C) = 1$ to get the vanishing of higher cohomology.

The results for $\mathcal{U}^*|_C$ follow using the defining sequence (26) restricted to C and the results for $\mathcal{U}^\perp|_C$ which we have just proved. \square

Lemma 2.38. *The following relations hold on Y for any conic C :*

$$\begin{aligned} \chi(\mathcal{U}|_C) &= 0 & \chi(V/\mathcal{U}|_C) &= 5 \\ h^0(\mathcal{U}|_C) &= 0 & h^0(V/\mathcal{U}|_C) &= 5 \\ h^1(\mathcal{U}|_C) &= 0 & h^1(V/\mathcal{U}|_C) &= 0 \end{aligned}$$

Proof. To compute $\chi(\mathcal{U}|_C)$ use Riemann–Roch.

To check $h^0(\mathcal{U}|_C) = 0$, assume $\mathcal{U}|_C$ has a section. By remark 2.31 we can think of it as $v \in V$. Then C embeds in the zero locus of some $v \in H^0(Y, V/\mathcal{U})$, which by 2.18 is either a line or a point.

The other results follow as in 2.37. \square

Lemma 2.39. *The following relations hold on Y for any conic C :*

$$\begin{aligned} \text{ext}^1(\mathcal{U}, \mathcal{I}_C) &= 0 \\ \text{hom}(\mathcal{U}, \mathcal{I}_C) &= 1 \end{aligned}$$

Proof. In order to prove $\text{ext}^1(\mathcal{U}, \mathcal{I}_C) = 0$, we will prove the surjectivity of

$$H^0(\mathcal{U}^*) \rightarrow H^0(\mathcal{U}^*|_C)$$

Take the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{U}^\perp & \longrightarrow & V^* \otimes \mathcal{O}_Y & \longrightarrow & \mathcal{U}^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{U}^\perp|_C & \longrightarrow & V^* \otimes \mathcal{O}_C & \longrightarrow & \mathcal{U}^*|_C & \longrightarrow & 0 \end{array} \quad (56)$$

After applying H^0 it becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & V^* & \xrightarrow{\cong} & H^0(\mathcal{U}^*) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathcal{U}^\perp|_C) & \longrightarrow & V^* & \xrightarrow{1_{V^*}} & H^0(\mathcal{U}^*|_C) & \longrightarrow & 0 \end{array}$$

where we have used $0 = H^1(\mathcal{U}^\perp|_C)$ from 2.37, and the rows are exact. Surjectivity follows by commutativity of the diagram.

To check $\text{hom}(\mathcal{U}, \mathcal{I}_C) = 1$ use the long exact sequence of

$$0 \rightarrow \mathcal{U}^*|_C \otimes \mathcal{I}_C \rightarrow \mathcal{U}^* \rightarrow \mathcal{U}^*|_C \rightarrow 0$$

together with $h^0(\mathcal{U}^*|_C) = 4$ from 2.37 and with $\text{ext}^1(\mathcal{U}, \mathcal{I}_C) = 0$, which we have just proved. \square

Now we want to use the above cohomological properties to recover the splitting type of the tautological bundles on conics.

Corollary 2.40. *The splitting types of $\mathcal{U}, V/\mathcal{U}, \mathcal{U}^\perp, \mathcal{U}^*$ on any smooth conic C are as follows:*

$$\begin{aligned} \text{ST}_C(\mathcal{U}) &= (-1, -1) \\ \text{ST}_C(V/\mathcal{U}) &= (0, 1, 1) \\ \text{ST}_C(\mathcal{U}^\perp) &= (-1, -1, 0) \\ \text{ST}_C(\mathcal{U}^*) &= (1, 1) \end{aligned}$$

Proof. The vector bundle embedding of \mathcal{U}^\perp into $V^* \otimes \mathcal{O}_Y$ implies that $\mathcal{U}^\perp|_C$ cannot have direct summands of strictly positive degree. Moreover, by lemma 2.37, we have $h^1(\mathcal{U}^\perp|_C) = 0$, which yields

$$\text{ST}_C(\mathcal{U}^\perp) = (-1, -1, 0)$$

as only possible splitting. Dualizing we also obtain the splitting type of V/\mathcal{U} .

To compute the splitting of \mathcal{U} , recall that by lemma 2.38 $\mathcal{U}|_C$ has no sections, which implies

$$\text{ST}_C(\mathcal{U}) = (-1, -1)$$

Dualize to get the splitting type of \mathcal{U}^* . \square

For any pair of points in \mathbb{P}^3 there is a line connecting them, and the space of lines through each point is irreducible. These properties are used in [OSS80] to prove that the splitting type of a stable bundle on a generic line has no gaps. As we would like to recover analogous results for instantons on Y , we prove now some properties for conics in Y which are analogous to properties of lines in \mathbb{P}^3 that we have just mentioned.

Recall from proposition 2.32 that the universal conic \mathcal{C} has two natural projections

$$\begin{array}{ccc} & \mathcal{C} & \\ q_A \swarrow & & \searrow q_Y \\ \mathbb{P}(V^*) & & Y \end{array}$$

which are naturally identified with the projections from $\mathbb{P}_Y(\mathcal{U}^\perp)$ and from $\mathbb{F}\ell(2, 4, V)|_Y$ to $\mathbb{P}(V^*)$ and Y .

This last fact has a few straightforward consequences. Denote by $\text{Hilb}_{Y,P}^{2t+1}$ the Hilbert scheme of conics passing through a point $P \in Y$ and by \mathcal{C}_P the universal conic passing through $P \in Y$. There is a cartesian square

$$\begin{array}{ccc} \text{Hilb}_{Y,P}^{2t+1} & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \square & \downarrow \\ \mathbb{P}(V^*) \times P & \longrightarrow & \mathbb{P}(V^*) \times Y \end{array}$$

as one can check by comparing the universal properties of the fiber product

$$\mathcal{C} \times_{\mathbb{P}(V^*) \times Y} (\mathbb{P}(V^*) \times P)$$

and of $\text{Hilb}_{Y,P}^{2t+1}$.

Corollary 2.41. *The Hilbert scheme of conics passing through a point $P \in Y_5$ is canonically isomorphic to $\mathbb{P}(U_P^\perp) \rightarrow \mathbb{P}(V^*)$ under the identification of $\mathbb{P}(V^*)$ and Hilb_Y^{2t+1} .*

Proof. By isomorphism (52) and by its definition, $\text{Hilb}_{Y,P}^{2t+1}$ is the fiber of $\mathbb{P}(\mathcal{U}^\perp)$, that is to say $\mathbb{P}(U_P^\perp) \rightarrow \mathbb{P}(V^*)$. \square

Remark 2.42. By corollary 2.41, condition (2) of theorem 1.26 holds for the universal family of conics in Y .

Another consequence of 2.32 is the following description of the space of conics through two points.

Corollary 2.43. *Given any two points $P, Q \in Y$ there is at least one conic C connecting them. Moreover the conic C is unique if and only if P, Q do not lie on a line.*

Proof. The Hilbert scheme of conics through P and Q is by definition the fiber product of $\mathbb{P}(U_P^\perp) \rightarrow \mathbb{P}(V^*)$ and $\mathbb{P}(U_Q^\perp) \rightarrow \mathbb{P}(V^*)$, that is to say $\mathbb{P}((U_P + U_Q)^\perp)$.

By corollary 2.23, $\mathbb{P}((U_P + U_Q)^\perp)$ is a point if and only if P and Q do not line on a line. In the case P and Q lie on a line, by the same corollary, there is a \mathbb{P}^1 of conics via P and Q . All of them contain the line through P and Q (and are therefore singular) as otherwise there would be a plane elliptic curve. As Y is cut by quadrics, if it contains a plane elliptic curve, it contains the whole plane, which is against Lefschetz hyperplane section theorem. \square

The following proposition describes the locus of singular conics.

Proposition 2.44. *The locus of singular conics in Y is the image of a degree 2 map from IQ to $\mathbb{P}(V)$. It is a divisor of degree 3 which is singular along a rational normal curve of degree 4 representing double lines.*

Proof. Over the space

$$IQ \subset \mathbb{P}(A) \times \mathbb{P}(A)$$

of intersecting pairs of lines there is a family of conics. This is clearly true for points corresponding to distinct lines, while it follows from the normal bundle description 2.27 in the case of points on the diagonal of $\mathbb{P}(A) \times \mathbb{P}(A)$.

This family of lines induces a regular map

$$IQ \rightarrow \mathbb{P}(V^*) \quad (57)$$

which by proposition 2.26 and corollary 2.41 has degree $(1, 1)$. The variety IQ has an involution interchanging the two factors, and the map (57) factors through this involution. Keeping this in mind and using projection formula, one can compute the degree of the image of (57) by intersecting it with a line in $\mathbb{P}(V^*)$.

Finally, one can check that the quotient by the involution of the map (57) is a closed embedding, so that its singular locus is the fixed locus of the involution, namely the curve of double lines. As there is only one orbit of dimension 1 for the action of SL_2 on $\mathbb{P}(V^*)$, the curve of double lines is a rational normal curve of degree 4 in $\mathbb{P}(V^*)$. \square

We will also discuss the intersection pairing of a line and a conic in lemma 3.5.

2.4 Cubics

The Beilinson spectral sequence approach works in the case of cubics in the same way as in the cases of lines and conics. As we have dealt with these other two cases in great detail in proposition 2.20 and proposition 2.32, we will only sketch the proofs in the case of cubics.

Remark 2.45. By cubic we mean a subscheme T of Y whose Hilbert polynomial is $3t + 1$. By corollary 1.39 such subschemes have no embedded points and satisfy $h^1(\mathcal{O}_T) = 0$.

Proposition 2.46. *The Hilbert scheme of cubics in Y is isomorphic to $\mathbb{G}r(2, V)$. The universal family of cubics Cub has a resolution*

$$0 \rightarrow \mathcal{U}(-1) \boxtimes \mathcal{O}_Y(-1) \rightarrow \mathcal{O}_{\mathbb{G}r}(-1) \boxtimes V/\mathcal{U}(-1) \rightarrow \mathcal{O}_{\mathbb{G}r} \boxtimes \mathcal{O}_Y \rightarrow \mathcal{O}_{Cub} \rightarrow 0$$

inside $\mathbb{G}r(2, V) \times Y$.

Proof. First we construct a family of cubics. Via Borel–Bott–Weil one can construct on $\mathbb{G}r(2, V) \times Y$ a complex

$$\mathcal{U}(-1) \boxtimes \mathcal{O}_Y(-1) \rightarrow \mathcal{O}_{\mathbb{G}r}(-1) \boxtimes V/\mathcal{U}(-1) \rightarrow \mathcal{O}_{\mathbb{G}r} \boxtimes \mathcal{O}_Y$$

where \mathcal{U} is as usual the rank 2 tautological bundle on $\mathbb{G}r(2, V)$. One can then prove that it has a unique non-vanishing cohomology and that its cohomology is flat over $\mathbb{G}r(2, V)$. The Chern character computation proves that it is a family of cubics in Y .

Given a family of cubics $T \subset S \times Y$, compute the relative Beilinson spectral sequence 1.18 with respect to the dual collections

$$\begin{aligned} &\langle \mathcal{O}_Y(-1), V/\mathcal{U}(1), \mathcal{U}, \mathcal{O}_Y \rangle \\ &\langle \mathcal{O}_Y, V/\mathcal{U}, \mathcal{U}^*, \mathcal{O}_Y(1) \rangle \end{aligned}$$

which one can obtain by dualizing collections (35) and (36).

From remark 2.45 we can deduce several other vanishings about the tautological bundles on Y restricted to T , which yield an exact sequence

$$0 \rightarrow \mathcal{M}_{\mathcal{O}(-1)} \boxtimes \mathcal{O}_Y(-1) \rightarrow \mathcal{L}_{V/\mathcal{U}(-1)} \boxtimes V/\mathcal{U}(-1) \rightarrow \mathcal{O}_S \boxtimes \mathcal{O}_Y \rightarrow \mathcal{O}_T \rightarrow 0$$

By corollary 1.20 the sheaf $\mathcal{M}_{\mathcal{O}(-1)}$ is a vector bundle of rank 2, while $\mathcal{L}_{V/\mathcal{U}(-1)}$ is a line bundle.

The proof that the two correspondences between families of cubics and maps to $\mathrm{Gr}(2, V)$ are inverse to each other is analogous to that of lines in proposition 2.20. \square

Remark 2.47. A more geometric construction for the universal family of cubics is the following. We will see later that there is a diagram

$$\begin{array}{ccc} & \mathbb{P}_Y(\mathcal{U}) \cong \mathrm{Bl}_{\sigma(A)}\mathbb{P}(V) & \\ \swarrow & & \searrow b \\ Y & & \mathbb{P}(V) \end{array}$$

By proposition 3.10, the Fourier–Mukai transform functor with kernel the structure sheaf $\mathrm{Bl}_{\sigma(A)}\mathbb{P}(V)$ of a line in $\mathbb{P}(V)$ is a cubic in Y . The Koszul complex for the universal line

$$\mathbb{P}(\mathcal{T}_2) \subset \mathrm{Gr}(2, V) \times \mathbb{P}(V)$$

yields a resolution

$$0 \rightarrow \mathcal{O}_{\mathrm{Gr}}(-1) \boxtimes \mathcal{U}(-1) \rightarrow V/\mathcal{T}_2 \boxtimes \mathcal{O}_Y(-1) \rightarrow \mathcal{O}_{\mathrm{Gr}} \boxtimes \mathcal{O}_Y \rightarrow \mathcal{O}_{\mathrm{Cub}} \rightarrow 0$$

One can also obtain the above resolution by using the relative Beilinson spectral sequence with respect to the collection $\langle \mathcal{O}_Y(-1), \mathcal{U}(-1), \mathcal{O}_Y, \mathcal{U} \rangle$.

3 More on the geometry of Y

In this section we rely on the results of section 2 to investigate further the geometry of Y . In 3.1 we construct an embedding of $\mathbb{P}(A)$ in $\mathrm{Gr}(3, V)$ and some exact sequences related to it. In 3.2 we show that the projectivization $\mathbb{P}_Y(\mathcal{U})$ is naturally isomorphic to a blow up of $\mathbb{P}(V)$. This leads to the construction of a family of linear sections of Y parametrized by $\mathrm{Gr}(3, V)$, which we describe in 3.3. Finally, in 3.4 we study a Fourier–Mukai transform in relation to the family of linear sections which we constructed.

3.1 Embeddings of $\mathbb{P}(A)$

First, recall that by definition 2.3 and by remark 2.4, the map σ is an embedding of $\mathbb{P}(A)$ in $\mathbb{P}(V)$ associating to a form $a \in A$ its unique kernel vector $a \wedge a \in \Lambda^4 V^* \cong V$. The image of $\mathbb{P}(A)$ via σ is denoted by $\sigma(A)$.

We can also embed $\mathbb{P}(A)$ in $\mathrm{Gr}(3, V)$ by mapping $a \in A$ to the kernel of the natural composition

$$V \longrightarrow A^* \otimes V^* \xrightarrow{\ker a} A^*$$

which we will denote by $A(\ker(a), -)^\perp$. We denote this embedding by κ and the tautological bundle on $\mathrm{Gr}(3, V)$ by \mathcal{K} . The proof that κ is an embedding is part of proposition 3.6 and is carried out with the help of lemma 3.1. The image of $\mathbb{P}(A)$ via κ is denoted by $\kappa(A)$.

Lemma 3.1. *Let \mathcal{K} be the tautological bundle on $\mathrm{Gr}(3, V)$. There is an exact sequence*

$$0 \rightarrow \kappa^* \mathcal{K} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(A)} \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(A)}(2) \rightarrow \mathcal{O}_{\mathbb{P}(A)}(3) \rightarrow 0 \quad (58)$$

The degree of κ is 3.

Proof. Define the middle map as the dual of the composition

$$A \otimes \mathcal{O}_{\mathbb{P}(A)}(-2) \xrightarrow{A \otimes \sigma} A \otimes V \otimes \mathcal{O}_{\mathbb{P}(A)} \xrightarrow{\mathrm{ev}} V^* \otimes \mathcal{O}_{\mathbb{P}(A)}$$

Using the fact 2.1 that each form in A has rank 4, the exactness of

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-3) \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(A)}(-2) \rightarrow V^* \otimes \mathcal{O}_{\mathbb{P}(A)}$$

can be checked pointwise, showing at the same time that the cokernel of

$$A \otimes \mathcal{O}_{\mathbb{P}(A)}(-2) \rightarrow V^* \otimes \mathcal{O}_{\mathbb{P}(A)}$$

is a vector bundle. The rank 3 quotient bundle of $V^* \otimes \mathcal{O}_{\mathbb{P}(A)}$ induces the map κ from $\mathbb{P}(A)$ to $\mathrm{Gr}(3, V)$. The degree of κ is the degree of $-c_1(\kappa^* \mathcal{K})$: by exact sequence (58) we find $\deg(\kappa) = 3$. \square

Under the canonical identification of $\mathrm{Gr}(3, V)$ with $\mathrm{Gr}(2, V^*)$, which is the space of pencils of conics in Y_5 , this embedding maps $a \in \mathbb{P}(A)$ into the unique pencil of conics sharing the line L_a as a component.

Lemma 3.2. *The scheme of conics containing a line $L := L_a$ is the line in $\mathbb{P}(V^*)$ corresponding to $\kappa(a) \in \mathrm{Gr}(2, V^*)$.*

Proof. Let L_a be the line corresponding to $a \in A$ and C_w the conic corresponding to $w \in V^*$ under the identifications of proposition 2.20 and 2.32. By corollary 2.23, L_a embeds in $\mathrm{Gr}(2, V)$ as $\mathbb{P}(\kappa(a)/\ker a)$, while by proposition 2.32 for any conic C_w we have

$$C_w = \mathbb{P}(A^\perp) \cap \mathrm{Gr}(2, \ker w) \subset \mathbb{P}(\Lambda^2 V)$$

It follows that L_a is contained in C_w if and only if $\kappa(a) \subset \ker w$, which means that w belongs to the line in $\mathrm{Gr}(2, V^*)$ corresponding to the 3-dimensional space $\kappa(a) \subset V$.

The schematic structure on the scheme of conics containing L is the reduced one as by corollary 2.41 it is the intersection of projective planes in $\mathbb{P}(V)$. \square

Lemma 3.3 also provides us with another description of $\kappa(A) \subset \mathbb{G}r(3, V)$. Note that a general projective plane $\mathbb{P}(K)$ in $\mathbb{P}(V)$ intersects $\kappa(A)$ in a finite number of points. The locus in $\mathbb{G}r(3, V)$ where this property fails is $\kappa(A)$.

Lemma 3.3. *The intersection $\mathbb{P}(K) \cap \sigma(A)$ inside $\mathbb{P}(V)$ is not of expected dimension if and only if $K \in \kappa(A) \subset \mathbb{G}r(3, V)$.*

Proof. Restrict the exact sequence

$$K^\perp \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(V)} \rightarrow \mathcal{O}_{\mathbb{P}(K)} \rightarrow 0$$

to $\mathbb{P}(A)$ via σ to find

$$K^\perp \otimes \mathcal{O}_{\mathbb{P}(A)}(-2) \rightarrow \mathcal{O}_{\mathbb{P}(A)} \rightarrow \mathcal{O}_{\mathbb{P}(K) \cap \sigma(A)} \rightarrow 0 \quad (59)$$

The intersection $\mathbb{P}(K) \cap \sigma(A)$ is not of expected dimension if and only if all conics in the pencil given by K^\perp share a common component L . This on the other hand happens if and only if the map (59) factors as

$$K^\perp \otimes \mathcal{O}_{\mathbb{P}(A)}(-2) \rightarrow \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{a^*} \mathcal{O}_{\mathbb{P}(A)}$$

that is to say that there is a commutative diagram

$$\begin{array}{ccc} K^\perp & \longrightarrow & V^* \\ \downarrow & & \downarrow \\ A^* & \xrightarrow{\otimes a^*} & S^2 A^* \end{array} \quad (60)$$

Note that $A^* \otimes a^*$ is never contained in V^* , as V^* is cut inside $S^2 A^*$ by the SL_2 -equivariant quadric $q \in S^2 A$ and q is non-degenerate. It follows that there is a regular map from $\mathbb{P}(A)$ to $\mathbb{G}r(2, V^*)$ which sends a^* to the fiber product of

$$A^* \xrightarrow{\otimes a^*} S^2 A^*$$

and of the canonical injection $V^* \rightarrow S^2 A^*$. Moreover, the locus of K^\perp having 1-dimensional intersection with $\sigma(A)$ is the image of $\mathbb{P}(A)$ under that map, so that it is at most 2-dimensional and it is irreducible.

We will prove that all planes in $\mathbb{P}(V)$ corresponding to points of $\kappa(A)$ have 1-dimensional intersection with $\sigma(A)$. Given a conic C , the set of lines which intersect it is a conic in $\mathbb{P}(A)$. More precisely, by lemma 3.5, the SL_2 -equivariant linear map from V^* to $S^2 A^*$ associates with a conic C the set of lines which intersect it, so that in order to show that $\mathbb{P}(K) \cap \mathbb{P}(A)$ is not finite it is enough to show that there are infinitely many lines intersecting all conics of the pencil $K^\perp \subset V^*$. But for $K \in \kappa(A)$ this last fact is true, as by lemma 3.2 all conics in the pencil contain a fixed line, which by proposition 2.26 intersects infinitely many other lines.

Finally, as $\kappa(A)$ has dimension 2, it is the whole locus of planes which intersect $\sigma(A)$ in a 1-dimensional scheme. \square

Remark 3.4. What we have actually proved is that if $K \in \kappa(A)$, then the intersection

$$\mathbb{P}(K) \cap \sigma(A) \subset \mathbb{P}(A)$$

is the extension of the structure sheaf of a line by that of a point. Furthermore, one can check that the point and the line are polar with respect to the intersection quadric \mathcal{Q} .

Denote by ILC the subvariety of $\mathbb{P}(A) \times \mathbb{P}(V^*)$ which is the closure of the locus of pairs L, C where L is a line and C is a conic intersecting L but not containing it.

Lemma 3.5. *The variety*

$$\text{ILC} \subset \mathbb{P}(A) \times \mathbb{P}(V^*)$$

of intersecting pairs of a line and a conic is cut by the SL_2 -equivariant

$$\mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(A)} \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}$$

Moreover, ILC is irreducible.

Proof. Given $a \in A$ and $w \in V^*$ let L_a be the corresponding line and C_w the corresponding conic under the identifications of propositions 2.20 and 2.32. Arguing as in lemma 3.2 one gets to

$$L_a \cap C_w = \mathbb{P}(\kappa(a)/\ker a) \cap \text{Gr}(2, \ker w) \subset \mathbb{P}(\Lambda^2 V)$$

As the intersection of $\kappa(a)$ and $\ker w$ inside V is always at least 2-dimensional, we find that L_a intersects C_w if and only if $\ker a \in \ker w$.

This last condition cuts a divisor inside $\mathbb{P}(A) \times \mathbb{P}(V^*)$ which is isomorphic to the following zero locus. Restrict the tautological

$$\mathcal{O}_{\mathbb{P}(V)}(-1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(V)} \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}$$

to $\mathbb{P}(A) \times \mathbb{P}(V)$ via $(\sigma \times Id)$ in order to find an SL_2 invariant divisor cut by

$$\mathcal{O}_{\mathbb{P}(A)}(-2) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(V)} \boxtimes \mathcal{O}_{\mathbb{P}(V^*)} \quad (61)$$

The above map vanishes at (a, w) if and only if $w(\ker a) = 0$.

Note that over $\mathbb{P}(A)$ the zero locus of (61) is the projectivization of the rank 4 vector bundle given by the kernel of the composition

$$V^* \otimes \mathcal{O}_{\mathbb{P}(A)} \rightarrow S^2 A^* \otimes \mathcal{O}_{\mathbb{P}(A)} \rightarrow \mathcal{O}_{\mathbb{P}(A)}(2)$$

so that it is clearly irreducible. It follows that the zero locus of (61) is the variety ILC of intersecting pairs of lines and conics. \square

We can also map $\mathbb{P}(A)$ into $\text{Gr}(3, V)$ by sending $a \in A$ to the embedded tangent space $T_{\sigma(a)}\sigma(A)$. We denote this map by τ and the image of $\mathbb{P}(A)$ via τ by $\tau(A)$.

Proposition 3.6. *The map κ is an embedding. There are exact sequences*

$$0 \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(V)}(-3) \rightarrow \Omega_{\mathbb{P}(V)}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(V)} \rightarrow \mathcal{O}_{\sigma(A)} \rightarrow 0 \quad (62)$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(V)}(-1) \oplus \mathcal{O}_{\mathbb{P}(V)}(-2) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow A^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \rightarrow \sigma_* \mathcal{O}_{\mathbb{P}(A)}(3) \rightarrow 0 \quad (63)$$

and

$$A \otimes \mathcal{K} \xrightarrow{d_1} V^* \otimes \mathcal{O}_{\mathbb{G}r} \rightarrow \kappa_* \mathcal{O}_{\mathbb{P}(A)}(2) \rightarrow 0 \quad (64)$$

Proof. First, note that the exact sequence (63) can also be written as

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(V)}(-2) \rightarrow \mathcal{T}_{\mathbb{P}(V)}(-1) \rightarrow A^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \rightarrow \sigma_* \mathcal{O}_{\mathbb{P}(A)}(3) \rightarrow 0$$

The above sequence can be obtained dualizing and twisting (62) as the dualizing sheaf of σ is the determinant of the normal bundle $\mathcal{N}_{\sigma(A)/\mathbb{P}(V)}$, which has odd degree. It follows that in order to prove (63) it is enough to prove (62).

We will prove the exactness of (63) by decomposing the ideal $\mathcal{I}_{\sigma(A)}$ of $\sigma(A)$ inside $\mathbb{P}(V)$ with respect to the full exceptional collection

$$\langle \mathcal{O}_{\mathbb{P}(V)}(-3), \Omega_{\mathbb{P}(V)}(-1), \mathcal{O}_{\mathbb{P}(V)}(-2), \mathcal{O}_{\mathbb{P}(V)}, \mathcal{O}_{\mathbb{P}(V)}(1) \rangle$$

As both $\mathcal{O}_{\mathbb{P}(V)}$ and $\mathcal{O}_{\sigma(A)}$ lie in the left orthogonal of $\mathcal{O}_{\mathbb{P}(V)}(1)$, so does $\mathcal{I}_{\sigma(A)}$. It is also clear that the ideal of $\sigma(A)$ is in the left orthogonal of $\mathcal{O}_{\mathbb{P}(V)}$.

In order to check that $\mathcal{I}_{\sigma(A)}$ is in the left orthogonal of $\mathcal{O}_{\mathbb{P}(V)}$ we can argue as follows. Assume there is a quadric in $\mathbb{P}(V)$ containing $\sigma(A)$, then its pullback to $\mathbb{P}(S^2 A)$ is a quadric containing the image of Ver_2 . As the ideal of Ver_2 does not contain quadrics, so does that of $\sigma(A)$. Moreover, the defining exact sequence for $\sigma(A)$ shows that $h^{>1}(\sigma(A)(2))$ vanishes and that $h^1 \neq 0$ if and only if $h^0 \neq 0$, so that we have proved that $H^\bullet(\sigma(A)(2)) = 0$.

Summing up we have proved that $\mathcal{I}_{\sigma(A)}$ belongs to

$$\langle \mathcal{O}_{\mathbb{P}(V)}(-3), \Omega_{\mathbb{P}(V)}(-1) \rangle$$

By Beilinson spectral sequence it follows that $\mathcal{O}_{\mathbb{P}(V)}(-3)$ and $\Omega_{\mathbb{P}(V)}(-1)$ can contribute non-trivially only in one degree. As their Chern characters are independent this leaves the unique possibility of a resolution

$$0 \rightarrow A \otimes \mathcal{O}_{\mathbb{P}(V)}(-3) \rightarrow \Omega_{\mathbb{P}(V)}(-1) \rightarrow \mathcal{I}_{\sigma(A)} \rightarrow 0$$

which can be extended to (62).

The next part of the statement is that κ is an embedding. First, note that as σ is an embedding also the product map (κ, σ) from $\mathbb{P}(A)$ to $\mathbb{G}r(3, V) \times \mathbb{P}(V)$ is an embedding. We are now going to prove that the image of (κ, σ) is the zero locus of the composition

$$A \otimes \mathcal{K} \boxtimes \mathcal{O}_{\mathbb{P}(V)}(-1) \rightarrow A \otimes V \otimes \mathcal{O}_{\mathbb{G}r(3, V)} \boxtimes V \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow \mathcal{O}_{\mathbb{G}r(3, V)} \boxtimes \mathcal{O}_{\mathbb{P}(V)} \quad (65)$$

of the tautological subbundles of $\mathbb{G}r(3, V)$ and $\mathbb{P}(V)$ with the natural pairing between A and $V \otimes V$. To prove it, we are going to show that the natural projection from

$\mathbb{G}r(3, V) \times \mathbb{P}(V)$ to $\mathbb{P}(V)$ induces an isomorphism between the zero locus Z of (65) and the degeneracy locus D of the SL_2 -equivariant

$$V \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow V \otimes V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \rightarrow A^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \quad (66)$$

The map from Z to D is constructed as follows. By definition of Z , the commutative diagram

$$\begin{array}{ccc} \mathcal{K} \boxtimes \mathcal{O}_{\mathbb{P}(V)} & & \\ \downarrow & \searrow & \\ V \otimes \mathcal{O}_{\mathbb{G}r} \boxtimes \mathcal{O}_{\mathbb{P}(V)} & \rightarrow & A^* \otimes \mathcal{O}_{\mathbb{G}r} \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1) \end{array} \quad (67)$$

restricts to

$$\begin{array}{ccc} \mathcal{K}|_Z & & \\ \downarrow & \searrow 0 & \\ V \otimes \mathcal{O}_Z & \rightarrow & A^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1)|_Z \end{array} \quad (68)$$

so that we have a rank 3 subbundle of the kernel of

$$V \otimes \mathcal{O}_Z \rightarrow A^* \otimes \mathcal{O}_{\mathbb{P}(V)}(1)|_Z$$

showing that the map from Z to $\mathbb{P}(V)$ factors via D .

The map from D to Z is constructed as follows. First note that over D there is an exact sequence

$$0 \rightarrow \text{Ker}_D \rightarrow V \otimes \mathcal{O}_D \rightarrow A^* \otimes \mathcal{O}_D(1) \rightarrow \text{Coker}_D \rightarrow 0$$

where $\mathcal{O}_D(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}(V)}(1)$. Note moreover that the rank of (66) never drops by 2 as the map σ is injective. It follows that Coker_D is a line bundle on D , so that Ker_D is a rank 3 subbundle of $V \otimes \mathcal{O}_D$. We now prove that the induced map from D to $\mathbb{G}r(3, V) \times \mathbb{P}(V)$ factors via Z . Over $\mathbb{G}r(3, V) \times \mathbb{P}(V)$ there is a diagram

$$\begin{array}{ccc} \mathcal{K} \boxtimes \mathcal{O}_{\mathbb{P}(V)} & \longrightarrow & V \otimes \mathcal{O}_{\mathbb{G}r} \boxtimes \mathcal{O}_{\mathbb{P}(V)} \\ & \searrow & \downarrow \\ & & A^* \otimes \mathcal{O}_{\mathbb{G}r} \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1) \end{array} \quad (69)$$

encoding both the tautological injection of $\mathbb{G}r(3, V)$ and that of $\mathbb{P}(V)$. By definition of the map from D to $\mathbb{G}r(3, V) \times \mathbb{P}(V)$, the diagram (69) restricts as

$$\begin{array}{ccc} \text{Ker}_D & \longrightarrow & V \otimes \mathcal{O}_D \\ & \searrow 0 & \downarrow \\ & & A^* \otimes \mathcal{O}_D(1) \end{array} \quad (70)$$

with the diagonal map from Ker_D to $A^* \otimes \mathcal{O}_D(1)$ being the zero map. By definition of Z , we have a map from D to Z .

The two maps between D and Z that we have just constructed are mutually inverse, as we are about to prove. There are commutative diagrams (69) (restricted to Z) and (70) which, under the two maps which we have just defined, pull back one to the other. As a consequence the map $Z \rightarrow D \rightarrow Z$ pulls the tautological injection of $\mathcal{K} \subset V \otimes \mathcal{O}_Z$ back to itself, so that the is Id_Z . Analogously, the map $D \rightarrow Z \rightarrow D$ pulls the universal Ker_D back to itself, so that it is Id_D .

Now that we know that the schematic zero-locus of (65) is $\mathbb{P}(A)$, we can pushforward $\text{Kosz}(A \otimes \mathcal{K} \boxtimes \mathcal{O}_{\mathbb{P}(V)}(-1))$ to $\text{Gr}(3, V)$ using proposition 1.28. Note that, by lemma 1.35, in order to prove that $\mathbb{P}(A)$ embeds into $\text{Gr}(3, V)$ via κ it is enough to check that κ has relative dimension 0 and that $\mathcal{O}_{\text{Gr}(3, V)} \rightarrow \kappa_* \mathcal{O}_{\mathbb{P}(A)}$ is surjective.

That κ is a finite map is clear for the following reason. As $\mathbb{P}(A)$ is proper, it is enough to show that κ has finite fibers. If this is not the case, κ contracts a divisor, which implies in turn that κ is a constant map. As $\ker a \in \kappa(a)$, if $\kappa(a)$ is constant then $\sigma(A) \subset \kappa(a)$, which is impossible.

The first spectral sequence for the derived pushforward of a complex has second page

$$R^i \pi_{\text{Gr}*} (\Lambda^{-j} (A \otimes \mathcal{K} \boxtimes \mathcal{O}_{\mathbb{P}(V)}(-1)))$$

It might have several non-vanishing entries, but the only ones interfering with the cohomology in degree 0 are those with $-j = i - 1, i, i + 1$. As $\dim \mathbb{P}(V) = 4$, i ranges from 0 to 4. As $\mathcal{O}_{\mathbb{P}(V)}(-i)$ is acyclic for $i \in [1, 4]$, the cohomology in degree 0 is the cokernel of

$$\Lambda^5 (A \otimes \mathcal{K}) \xrightarrow{d_5} \mathcal{O}_{\text{Gr}}$$

Note that the composition (65) cuts a 2-dimensional scheme inside the 10-dimensional scheme $\text{Gr}(3, V) \times \mathbb{P}(V)$. It follows by proposition 1.28 and by the fact that a smooth subvariety of a smooth variety is a local complete intersection that $\text{Kosz}(A \otimes \mathcal{K} \boxtimes \mathcal{O}_{\mathbb{P}(V)}(-1))$ has non vanishing cohomology only in degrees -1 and 0 . If we denote by $\mathcal{L}_{\mathbb{P}(A)}$ the excess bundle from proposition 1.28, then the second page of the other spectral sequence for the pushforward of $\text{Kosz}(A \otimes \mathcal{K} \boxtimes \mathcal{O}_{\mathbb{P}(V)}(-1))$ is

$$R^i \pi_{\text{Gr}*} (\mathcal{L}_{\mathbb{P}(A)}) \quad R^i \pi_{\text{Gr}*} (\mathcal{O}_{\mathbb{P}(A)})$$

As we have already pointed out in this proof, κ is finite, so that the only non-vanishing entries are

$$\kappa_* (\mathcal{L}_{\mathbb{P}(A)}) \quad \kappa_* (\mathcal{O}_{\mathbb{P}(A)})$$

with the spectral sequence degenerating at this page. As a consequence the sequence

$$\Lambda^5 (A \otimes \mathcal{K}) \xrightarrow{d_5} \mathcal{O}_{\text{Gr}} \rightarrow \kappa_* \mathcal{O}_{\mathbb{P}(A)} \rightarrow 0$$

is exact and κ is an embedding.

The last part of the statement is the sequence (64). Take the Koszul complex of (65), twist it by $\mathcal{O}_{\mathbb{P}(V)}(1)$ and push it forward to $\mathbb{P}(V)$. As in the previous step of the

proof, we will find two spectral sequences: by comparing them and by Künneth formula we find an exact sequence

$$A \otimes \mathcal{K} \xrightarrow{d_1} V^* \otimes \mathcal{O}_{\mathbb{G}\mathbb{r}} \rightarrow \kappa_* \mathcal{O}_{\mathbb{P}(A)}(2) \rightarrow 0$$

which finally proves (64). □

3.2 Identification with blow up

The map $\mathcal{U} \rightarrow V \otimes \mathcal{O}_Y$ induces a diagram

$$\begin{array}{ccc} & \mathbb{P}_Y(\mathcal{U}) & \\ p_Y \swarrow & & \searrow p_V \\ Y & & \mathbb{P}(V) \end{array}$$

In the next lemma we will prove that the above diagram is isomorphic to another natural correspondence between Y and $\mathbb{P}(V)$.

Proposition 3.7. *There is an isomorphism*

$$f : \mathbb{P}_Y(\mathcal{U}) \rightarrow \mathrm{Bl}_{\sigma(A)} \mathbb{P}(V)$$

commuting with the natural projections to $\mathbb{P}(V)$.

Remark 3.8. As a consequence of lemma 3.7, from now on we will denote also the projections in

$$\begin{array}{ccc} & \mathrm{Bl}_{\sigma(A)} \mathbb{P}(V) & \\ p_Y \swarrow & & \searrow p_V \\ Y & & \mathbb{P}(V) \end{array} \tag{71}$$

by p_Y and p_V .

Proof. Over the ambient Grassmannian $\mathbb{G}\mathbb{r}(2, V)$ there are natural identifications

$$\mathbb{P}_{\mathbb{G}\mathbb{r}(2, V)}(\mathcal{U}) \cong \mathbb{F}\mathbb{L}(1, 2, V) \cong \mathbb{P}_{\mathbb{P}(V)}(\mathcal{T}_{\mathbb{P}(V)}(-2))$$

relative to $\mathbb{G}\mathbb{r}(2, V)$. The last space is by definition

$$\mathrm{Proj}_{\mathbb{P}(V)}(\mathrm{Sym}^\bullet(\Omega_{\mathbb{P}(V)}(2)))$$

When we restrict to Y , we are intersecting the above space with

$$\mathrm{Proj}_{\mathbb{P}(V)}(\mathrm{Sym}^\bullet(A \otimes \mathcal{O}_{\mathbb{P}(V)}))$$

inside $\mathbb{P}(V) \times \mathbb{P}(\Lambda^2 V)$.

The result is the projectivization of the symmetric algebra of

$$\text{coker}(A \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow \Omega_{\mathbb{P}(V)}(2)) \quad (72)$$

By exact sequence (62) and SL_2 -equivariance of the whole construction, the cokernel (72) is isomorphic to a twist of the ideal of $\sigma(A)$ in $\mathbb{P}(V)$, so that its **Proj** is by definition the blowup of $\mathbb{P}(V)$ in $\sigma(A)$. \square

Definition 3.9. Denote by H the pullback of the ample generator from Y , by h the pullback of the ample generator from $\mathbb{P}(V)$, and by E the exceptional divisor of the blow-up.

Since the Picard number of $\mathbb{P}_Y(\mathcal{U})$ is 2, we look for a relation between H , h and E . We will use the relative tautological sequence

$$0 \rightarrow \mathcal{O}(-h) \rightarrow p_Y^* \mathcal{U} \rightarrow \mathcal{O}(H-h) \rightarrow 0 \quad (73)$$

on $\mathbb{P}_Y(\mathcal{U})$.

Proposition 3.10. *The relative canonical class of $\mathbb{P}_Y(\mathcal{U})$ over Y is $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h-E)$. Moreover,*

$$H = 3h - E \quad (74)$$

Proof. Compute the canonical class of $\mathbb{P}_Y(\mathcal{U})$ in two different ways. On one hand, using the blow-up description, it is $-5h+E$. On the other hand, on $\mathbb{P}_Y(\mathcal{U})$ there's a tautological injection

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-h) \rightarrow p_Y^* \mathcal{U} \quad (75)$$

which gives as relative cotangent bundle $\omega_{p_Y} = \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(H-2h)$. Together with $\omega_Y = -2H$ this gives $\omega_{\mathbb{P}(\mathcal{U})} = -H-2h$, so that in the end $H = 3h - E$. \square

We can use propositions 3.7 and 3.10 to describe how lines transform via the correspondence $\mathbb{P}_Y(\mathcal{U})$.

Proposition 3.11. *Let L be a line in $\mathbb{P}(V)$, \tilde{L} be its strict transform in $\mathbb{P}(\mathcal{U})$ and $L^+ = p_Y(\tilde{L})$. Then*

$$3 - \ell(\sigma(A) \cap L) = \deg(L^+)$$

where ℓ denotes the length of the structure sheaf of a scheme.

Proof. First, note that it makes sense to talk about length, as there is no line inside $\mathbb{P}(A)$. Then, by projection formula $\deg(L^+) = H \cdot \tilde{L}$. Moreover $H = 3h - E$ by equation (74), and again by projection formula $l(\sigma(A) \cap L) = l(E \cap \tilde{L})$. \square

Given a point $y \in Y$, its transform $p_Y p_Y^{-1}(y)$ in $\mathbb{P}(V)$ is a line. More precisely, the projective bundle $\mathbb{P}_Y(\mathcal{U})$ gives a family of lines in $\mathbb{P}(V)$ parametrized by Y . By

proposition 3.11 all lines $p_V p_Y^{-1}(y)$ are trisecants to $\sigma(A)$, so that we choose the following notation for the induced map from Y to $\mathbb{G}r(2, V)$:

$$\begin{aligned} \text{tri} : Y &\longrightarrow \text{Tri} \subset \mathbb{G}r(2, V) \\ y &\longmapsto p_V p_Y^{-1}(y) \end{aligned}$$

where Tri is the space of trisecant lines to $\sigma(A) \subset \mathbb{P}(V)$. The following corollary was already known to Castelnuovo [Cas91].

Corollary 3.12. *The map tri is an isomorphism from Y to the space Tri of lines in $\mathbb{P}(V)$ which are trisecant to $\sigma(A)$.*

Proof. The map tri is defined by the family $\mathbb{P}_Y(\mathcal{U})$, so that it is regular. For a line L in $\mathbb{P}(V)$, let \tilde{L} be its strict transform in $\text{Bl}_{\sigma(A)}(\mathbb{P}(V))$. Define the inverse of tri as

$$\begin{aligned} \text{tri}^{-1} : \text{Tri} &\longrightarrow Y \\ L &\longmapsto p_Y(\tilde{L}) \end{aligned}$$

Note that tri^{-1} is well defined on Tri by proposition 3.11. Also tri^{-1} can be defined in families, and it is easy to check that the two morphisms are inverse to each other. \square

3.3 A map from $\mathbb{G}r(3, V)$ to $\mathbb{P}(\Lambda^2 V^*/A)$

Given $\mathbb{P}(K) \subset \mathbb{P}(V)$, $\dim(K) = 3$, there are two natural ways to obtain a point in $\mathbb{P}(\Lambda^2 V^*/A)$. We define a map α in (76) and we prove in proposition 3.16 that there is another description for it.

Definition 3.13. The map α is the composition of

$$\mathbb{G}r(3, V) \xrightarrow{\cong} \mathbb{G}r(2, V^*) \hookrightarrow \mathbb{P}(\Lambda^2 V^*) \dashrightarrow \mathbb{P}(\Lambda^2 V^*/A) \quad (76)$$

where the second map is the Plücker embedding and the third is the linear projection from $\mathbb{P}(A)$.

The map α is regular on $\mathbb{G}r(2, V^*)$, as $\mathbb{P}(A) \cap \mathbb{G}r(2, V^*) = \emptyset$ by lemma 2.1. It provides us with a family of linear sections of Y parametrized by $\mathbb{G}r(3, V)$. As we will often use the total preimage $p_V^{-1}(\mathbb{P}(K))$, we introduce the following notation.

Definition 3.14. Recall that \mathcal{K} denotes the tautological bundle over $\mathbb{G}r(3, V)$. The scheme \mathcal{S} is the fiber product

$$\begin{array}{ccc} \mathcal{S} & \hookrightarrow & \mathbb{G}r(3, V) \times \mathbb{P}_Y(\mathcal{U}) \\ \downarrow & \square & \downarrow \text{Id}_{\mathbb{G}r} \times p_V \\ \mathbb{P}_{\mathbb{G}r(3, V)}(\mathcal{K}) & \hookrightarrow & \mathbb{G}r(3, V) \times \mathbb{P}(V) \end{array} \quad (77)$$

We denote the fiber of \mathcal{S} over a point $K \subset V$ of $\mathbb{G}r(3, V)$ by S_K , and its image $p_Y(S_K)$ by Y_K .

Remark 3.15. We will see in proposition 3.16 that Y_K is a linear section of Y and that, under the canonical identification of the space of linear sections of Y with $\mathbb{P}(\Lambda^2 V^*/A)$, Y_K corresponds to $\alpha(K)$.

Denote by Φ_{Bl} the Fourier–Mukai functor with kernel $\mathcal{O}_{\text{Bl}_{\sigma(A)}\mathbb{P}(V)}$, that is to say

$$\Phi_{\text{Bl}} := R p_{V*} L p_Y^* : D^b(\mathbb{P}(V)) \rightarrow D^b(Y_5) \quad (78)$$

where p_Y and p_V are the projections from $\text{Bl}_{\sigma(A)}\mathbb{P}(V)$ to Y and V , as defined in diagram (71). For $K \subset V$ a 3-dimensional subspace, proposition 3.16 shows that $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)})$ is the structure sheaf of a linear section of Y .

Proposition 3.16. *Let K be a 3-dimensional subspace of V . In the notation of definition 3.14 and remark 3.15*

$$\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}) = \mathcal{O}_{Y_K}$$

Moreover, Y_K is the linear section of Y corresponding to $\alpha(K)$ under the natural identification of the space of linear sections of Y with $\mathbb{P}(\Lambda^2 V^/A)$.*

Proof. First, note that the fiber product (77) has expected dimension because for each K we have $\mathbb{P}(K) \notin \sigma(A)$ and because the relative dimension of p_V over $\mathbb{P}(V)$ is 1. It follows by lemma 1.33 that the diagram is Tor-independent and that $L p_V^* \mathcal{O}_{\mathbb{P}(K)} \cong \mathcal{O}_S$.

Now, let $\mathcal{K} \subset V \otimes \mathcal{O}_{\mathbb{G}\text{r}(3,V)}$ be the tautological injection. Over $\mathbb{P}_{\mathbb{G}\text{r}}(V \otimes \mathcal{O}_{\mathbb{G}\text{r}})$ there is a universal Koszul resolution

$$\mathcal{O}_{\mathbb{P}(K)} \cong \{ \Lambda^2 \mathcal{K}^\perp(-2h) \rightarrow \mathcal{K}^\perp(-h) \rightarrow \mathcal{O}_{\mathbb{P}(V) \times \mathbb{G}\text{r}(3,V)} \}$$

for the structure sheaf of $\mathbb{P}_{\mathbb{G}\text{r}(3,V)}(\mathcal{K})$.

Pulling back via $\text{Id}_{\mathbb{G}\text{r}} \times p_V$ we get

$$\mathcal{O}_S \cong \{ \Lambda^2 \mathcal{K}^\perp(-2h) \rightarrow \mathcal{K}^\perp(-h) \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U}) \times \mathbb{G}\text{r}(3,V)} \} \quad (79)$$

As $\mathcal{O}(h)$ is the relative $\mathcal{O}(1)$ of p_Y , we can use this resolution to compute $\Phi(\mathcal{O}_{\mathbb{P}(K)})$ as the limit of a spectral sequence. The only page with non vanishing differentials is

$$\begin{array}{ccccc} \det(\mathcal{K}^\perp) \boxtimes \mathcal{O}_Y(-H) & 0 & & 0 & \\ & \searrow d_2 & & & \\ 0 & 0 & & \mathcal{O}_{\mathbb{G}\text{r}(3,V) \times Y} & \end{array}$$

where in order to compute the entries we have used relative Serre duality for the projection from $\mathbb{G}\text{r}(3,V) \times \mathbb{P}_Y(\mathcal{U})$ to $\mathbb{G}\text{r}(3,V) \times Y$ and the fact that $\det(\mathcal{U}) = \mathcal{O}(-H)$.

Twist d_2 by $\mathcal{O}_Y(H)$ and push it forward to $\mathbb{G}\text{r}(3,V)$ to obtain

$$\det(\mathcal{K}) \rightarrow (\Lambda^2 V^*/A) \otimes \mathcal{O}_{\mathbb{G}\text{r}(3,V)} \quad (80)$$

where we have substituted $\det(\mathcal{K}) = \det(\mathcal{K}^\perp)$ and $H^0(\mathcal{O}_Y(1)) = (\Lambda^2 V^*/A)$.

We will now check that the quotient of (80) is locally free by base-changing to points of $\mathrm{Gr}(3, V)$. Let K be a 3-dimensional subspace of V , then diagram (77) restricts to

$$\begin{array}{ccc} S & \hookrightarrow & \mathbb{P}_Y(\mathcal{U}) \\ \downarrow & \square & \downarrow p_V \\ \mathbb{P}(K) & \hookrightarrow & \mathbb{P}(V) \end{array} \quad (81)$$

Once more, the fiber product has expected dimension, so that $Lp_V^* \mathcal{O}_{\mathbb{P}(K)} \cong p_V^* \mathcal{O}_{\mathbb{P}(K)}$ and $p_V^* \mathcal{O}_{\mathbb{P}(K)} \cong \mathcal{O}_S$. As a consequence there is a quasi isomorphism between

$$\mathcal{O}_Y(-H) \xrightarrow{d_2|_{[K] \times Y}} \mathcal{O}_Y \quad (82)$$

and $Rp_{Y*} \mathcal{O}_S$. If d_2 restricted to $[K] \times Y$ were 0, then there would be a nontrivial $R^{-1}p_{Y*} \mathcal{O}_S$. It follows that $d_2|_{[K] \times Y} \neq 0$ and therefore (80) is an injection of vector bundles.

Finally, (80) induces a regular map from $\mathrm{Gr}(3, V)$ to $\mathbb{P}(\Lambda^2 V^*/A)$ which is SL_2 -equivariant as Φ_{Bl} is SL_2 -equivariant. We claim that this map is α . By SL_2 -equivariance, it is enough that the degree of the map which associates $\Phi_{\mathrm{Bl}}(\mathcal{O}_K)$ with K is 1. This last fact is a straightforward consequence of (80) and of $\det(\mathcal{K}) = \mathcal{O}_{\mathrm{Gr}(3, V)}(-1)$. \square

We are now going to describe the preimage via α of the branching locus of α in $\mathbb{P}(\Lambda^2 V^*/A)$. If we think of $\mathbb{P}(\Lambda^2 V^*/A)$ as of the space of hyperplane sections of Y , by projective duality the branching locus of α corresponds to singular hyperplane sections of Y . It follows that the preimage via α of the branching locus of α parametrizes subspaces $K \subset V$ such that the transform $p_V^* p_{Y*} \mathcal{O}_{\mathbb{P}(K)}$ is the structure sheaf of a singular hyperplane sections.

Proposition 3.18 is stated in terms of two subschemes of $\mathrm{Gr}(2, V)$: we will denote them by D_{tri} and by D_{fat} .

Definition 3.17. The closed subscheme $D_{tri} \subset \mathrm{Gr}(3, V)$ is the locus of $K \subset V$ which contain a trisecant to $\sigma(A)$. The closed subscheme $D_{fat} \subset \mathrm{Gr}(3, V)$ is the locus of $K \subset V$ such that the intersection with $\sigma(A)$ has a non-trivial tangent space.

Note that D_{fat} contains all $K \subset V$ such that the intersection is a non-reduced subscheme or is positive dimensional. We have already seen in lemma 3.3 that $\kappa(A) \subset D_{fat}$. In the proof of 3.3 we have seen that if $K \in \kappa(A)$, then the intersection is the union of a conic and a point lying on the plane spanned by the conic: it follows that $\kappa(A) \subset D_{tri}$.

Proposition 3.18. *The ramification locus of α is linearly equivalent to $\mathcal{O}_{\mathrm{Gr}(3, V)}(2)$. D_{tri} and D_{fat} are irreducible divisors in $\mathrm{Gr}(3, V)$. Their union is the preimage via α of the branching locus.*

Proof. The ramification divisor is linearly equivalent to the relative canonical bundle of α . As α is a linear projection we have

$$\alpha^* \mathcal{O}_{\mathbb{P}(\Lambda^2 V^*/A)}(1) = \mathcal{O}_{\mathrm{Gr}(3, V)}(1)$$

It follows that the relative canonical bundle of α is $\mathcal{O}_{\text{Gr}(3,V)}(2)$.

Let $K \subset V$ be a vector subspace of dimension 3. The next step we are going to prove is the following: whenever $K \notin D_{tri} \cup D_{fat}$, the linear section of Y corresponding to $\alpha(K)$ is regular.

By proposition 3.16, the structure sheaf of the linear section corresponding to $\alpha(K)$ is $\mathcal{O}_{Y_K} = \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)})$. By proposition 3.10, the projection p_Y from $\text{Bl}_{\sigma(A)}\mathbb{P}(V)$ to Y is given by the complete linear system $3h - E$. As $K \notin D_{fat}$, the total preimage S_K of $\mathbb{P}(K)$ under p_Y is the blow-up of $\mathbb{P}(K)$ in 4 distinct points. As $K \notin D_{tri}$ any triple of blown up points does not lie on a line. It is easy to check that in this case $3h - E$ is very ample on S_K . For example we can first check that $3h - E$ does not contract curves and after that, using $Rp_{Y*}\mathcal{O}_{S_K} = \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}) = \mathcal{O}_{Y_K}$, we can conclude that

$$p_Y : S_K \rightarrow Y_K$$

is an isomorphism on its image by lemma 1.35. It follows that the preimage via α of the branch locus is $D_{tri} \cup D_{fat}$.

Now we prove that D_{tri} and D_{fat} are distinct irreducible divisors in $\text{Gr}(3,V)$. First, we check that they are distinct.

Given a point y in the open orbit of Y , by corollaries 2.24 and 3.12 we have that $\text{tri}(y)$ is a trisecant to $\sigma(A)$ intersecting it in 3 distinct points, corresponding to the 3 distinct lines L_1, L_2, L_3 through y . Call $K_{a,y}$ the plane spanned by a and $\text{tri}(y)$, where we choose $a \in \sigma(A)$ such that L_a does not intersect the 3 lines L_1, L_2, L_3 through y (i.e. the lines in $\mathbb{P}(V)$ through a and any point in $\text{tri}(y) \cap \sigma(A)$ are not trisecants).

If $\mathbb{P}(K_{a,y})$ contains tangent vectors to $\sigma(A)$, then its intersection with $\sigma(A)$ has length at least 5 and is therefore 1-dimensional. In this case remark 3.4 says that $\mathbb{P}(K_{a,y}) \cap \sigma(A)$ is the union of a point and of a conic (in $\mathbb{P}(K_{a,y})$). But then any line (in $\mathbb{P}(K_{a,y})$) through the point is a trisecant, against our choice of a and $\text{tri}(y) \cap \sigma(A)$.

To show that D_{tri} is an irreducible divisor, construct the universal space \widetilde{D}_{tri} of pairs given by a plane $\mathbb{P}(K)$ in $\mathbb{P}(V)$ and a trisecant to $\sigma(A)$ contained in the plane. As Y is the space of trisecants to $\sigma(A)$ (see corollary 3.12), there is a diagram

$$\begin{array}{ccc} & \widetilde{D}_{tri} & \\ \swarrow & & \searrow p_{tri} \\ Y & & D_{tri} \end{array}$$

Moreover, for each trisecant line to $\sigma(A)$ there is a \mathbb{P}^2 of planes in $\mathbb{P}(V)$ containing it. More precisely, \widetilde{D}_{tri} is $\mathbb{P}_Y(V/\mathcal{U})$, which is clearly irreducible and 5-dimensional.

Finally, the generic fiber of p_{tri} is one point, as we are about to show. Take a plane $\mathbb{P}(K)$ in $\mathbb{P}(V)$ intersecting $\sigma(A)$ with expected dimension and containing 2 trisecants. Each one cuts a scheme of length 3 inside $\mathbb{P}(K) \cap \sigma(A)$, which has length 4. It follows that the 2 trisecants intersect in a scheme of length at least 2 and that, being lines, they coincide. As a consequence D_{tri} is 5-dimensional and irreducible, as it is the image under the generically finite map p_{tri} of a 5-dimensional irreducible scheme.

As for D_{fat} , construct the universal space $\widetilde{D_{fat}}$ of pairs given by a plane $\mathbb{P}(K)$ in $\mathbb{P}(V)$ and a tangent direction contained in the intersection $\mathbb{P}(K) \cap \sigma(A)$. The space of tangent directions to $\sigma(A)$ is the projectivization of the tangent bundle to $\mathbb{P}(A)$, which we denote by $\mathbb{P}_{\mathbb{P}(A)}(\mathcal{T}_{\mathbb{P}(A)})$, so that there is a diagram

$$\begin{array}{ccc} & \widetilde{D_{fat}} & \\ p_{fat} \swarrow & & \searrow \\ D_{fat} & & \mathbb{P}_{\mathbb{P}(A)}(\mathcal{T}_{\mathbb{P}(A)}) \end{array}$$

The space $\widetilde{D_{fat}}$ is a \mathbb{P}^2 -fibration over $\mathbb{P}_{\mathbb{P}(A)}(\mathcal{T}_{\mathbb{P}(A)})$, so that it is 5-dimensional and irreducible.

Moreover the fiber of p_{fat} over the generic point of D_{fat} is finite for the following reason. Assume a plane $\mathbb{P}(K)$ has 0-dimensional intersection with $\sigma(A)$, then if $\mathbb{P}(K)$ contains a 1-dimensional family of tangent directions to $\sigma(A)$ there is a point a in $\mathbb{P}(K) \cap \sigma(A)$ such that $\mathbb{P}(K)$ contains a 1-dimensional family of tangent directions to $\sigma(A)$ at a . As $\sigma(A)$ has dimension 2 and is smooth, this is enough to deduce that $\mathbb{P}(K)$ is the tangent space to $\sigma(A)$ at a . The locus of tangent planes to $\sigma(A)$ and tangent directions is a 3-dimensional subvariety of $\widetilde{D_{tri}}$, so that the generic point of $\widetilde{D_{fat}}$ is not contained in it. It follows that D_{fat} is 5-dimensional and irreducible as it is the image under the generically finite map p_{fat} of a 5-dimensional irreducible scheme. \square

Remark 3.19. One can also check that the branch locus has degree 10, and that the components of the fiber over its the generic point have multiplicities $(2, 1, 1, 1)$, with the non-reduced fiber, which corresponds to a ramification point, belonging to D_{tri} .

We will use the description of the ramification locus of proposition 3.18 to prove the following result about smooth linear sections containing a fixed line.

Lemma 3.20. *For any line L and for the generic plane $\mathbb{P}(K) \subset \mathbb{P}(V)$ containing $\sigma(L)$, the linear section Y_K is smooth. Moreover, for any line L in Y , the generic hyperplane section $Y_K \subset Y$ containing L is smooth.*

Proof. The second assertion clearly follows from the first as if $\sigma(L) \in \mathbb{P}(K)$ then $L \subset Y_K$.

By proposition 3.18 the set of planes $\mathbb{P}(K) \subset \mathbb{P}(V)$ inducing singular sections of Y is the union of D_{tri} and D_{fat} . For any L there is a 4-dimensional family of planes in $\mathbb{P}(V)$ containing $\sigma(L)$, namely $\mathbb{G}r(2, V/\sigma(L))$ and by smoothness of $\sigma(A)$ the generic one is not in D_{fat} .

We still need to prove that the generic $\mathbb{P}(K)$ through $\sigma(L)$ does not contain any trisecant. First, we prove that the generic bisecant through $\sigma(L)$ is not a trisecant. This is true as trisecants through $\sigma(L)$ correspond via the maps tri and tri^{-1} of corollary 3.12 to points of L , so that there is a one dimensional family of trisecants through $\sigma(L)$ (corresponding to points in L).

Now, choose two other points $L_1, L_2 \in \mathbb{P}(A)$ such that the three bisecants connecting $\sigma(L)$, $\sigma(L_1)$ and $\sigma(L_2)$ are not trisecants and consider the plane $\mathbb{P}(K)$ spanned by $\sigma(L)$, $\sigma(L_1)$ and $\sigma(L_2)$. Then clearly $\mathbb{P}(K) \notin D_{tri}$. \square

Lemma 3.21. $\kappa(A)$ is contained in D_{tri} .

Proof. By lemma 3.3 if $K \in \kappa(A)$ then the intersection of $\mathbb{P}(K)$ with $\sigma(A)$ is the union of a conic and a point (possibly lying on the conic). It follows that any line in $\mathbb{P}(K)$ passing through the extra point is a trisecant contained in $\mathbb{P}(K)$. \square

3.4 Linear sections of Y and the Fourier–Mukai transform Φ_{Bl}

In proposition 3.16 we have given an interpretation of the map α in terms of a Fourier–Mukai transform functor. In this section we use the Fourier–Mukai transform

$$\Phi_{Bl} := R p_{Y*} L p_V^* : D^b(\mathbb{P}(V)) \rightarrow D^b(Y)$$

to relate the incidence properties of planes in $\mathbb{P}(V)$ and $\sigma(A)$ with the properties of certain sheaves supported on linear sections of Y .

Recall from definition 3.14 that for any $K \in \mathbb{G}r(3, V)$ we denote by S_K the total preimage of $\mathbb{P}(K)$ via p_V and that we denote by Y_K the image of S_K via p_Y , as in the following diagram.

$$\begin{array}{ccccc} \mathbb{P}(K) & \xleftarrow{p_K} & S_K & \xrightarrow{p_{Y_K}} & Y_K \\ i_K \downarrow & & \square & & \downarrow i_{Y_K} \\ \mathbb{P}(V) & \xleftarrow[p_V]{} & \mathbb{P}_Y(\mathcal{U}) & \xrightarrow[p_Y]{} & Y \end{array} \quad (83)$$

We will often need to compute the cohomology of sheaves on S_K which come by pullback from $\mathbb{P}(K)$. This can be easily done via projection formula, provided one has already computed $R p_{K*} L p_K^* \mathcal{O}_{\mathbb{P}(K)}$

Lemma 3.22. The projection $p_K : S_K \rightarrow \mathbb{P}(K)$ is such that the canonical map

$$\mathcal{O}_{\mathbb{P}(K)} \rightarrow R p_{K*} \mathcal{O}_{S_K} \quad (84)$$

is an isomorphism.

Proof. The map p_K is defined via the following cartesian diagram

$$\begin{array}{ccc} S_K & \xrightarrow{i_S} & Bl_{\mathbb{P}(A)} \mathbb{P}(V) \\ p_K \downarrow & & \downarrow p_V \\ \mathbb{P}(K) & \xrightarrow{i_K} & \mathbb{P}(V) \end{array} \quad (85)$$

The fiber product in the cartesian diagram (85) has expected dimension and $\mathbb{P}(K)$, $\mathbb{P}(V)$, $\mathbb{P}_Y(\mathcal{U})$ are smooth. It follows by lemma 1.33 that the diagram (85) is Tor-independent, so that the base change map is an isomorphism. As a consequence

$$R p_{K*} \mathcal{O}_{S_K} \cong R p_{K*} L i_S^* \mathcal{O}_{Bl_{\mathbb{P}(A)} \mathbb{P}(V)} \cong L i_K^* R p_{V*} \mathcal{O}_{Bl_{\mathbb{P}(A)} \mathbb{P}(V)} \quad (86)$$

As the $R p_{V*} \mathcal{O}_{Bl_{\mathbb{P}(A)} \mathbb{P}(V)}$ is $\mathcal{O}_{\mathbb{P}(V)}$, so that the right hand side of (86) is $\mathcal{O}_{\mathbb{P}(K)}$. \square

The following lemma holds.

Lemma 3.23. *For any $K \subset V$ 3-dimensional vector subspace, $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ is a pure object concentrated in degree 0. Moreover it is equivalent to*

$$\{\mathcal{O}_Y \rightarrow \mathcal{U}^{*\oplus 2} \rightarrow S^2\mathcal{U}^*\} \quad (87)$$

Proof. By arguing as in 3.16 we get

$$p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h) \cong \{\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})} \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(2h)\}$$

As $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h)$ is the relative $\mathcal{O}(1)$ of p_Y , we can use this resolution to get $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ as the limit of a spectral sequence. The page with horizontal differentials is the last one with non-trivial maps and is

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \mathcal{O}_Y & \longrightarrow & \mathcal{U}^{*\oplus 2} & \longrightarrow & S^2\mathcal{U}^* \end{array}$$

as p_Y has relative dimension 1. This shows that $R^1 p_{Y*} \mathcal{O}_{\mathbb{P}(K)}(2h) = 0$ (i.e. that $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ is equivalent to a sheaf) and that the sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{U}^{*\oplus 2} \rightarrow S^2\mathcal{U}^* \rightarrow \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)) \rightarrow 0 \quad (88)$$

is exact. \square

We can use the resolution for $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ to compute several things about it. Recall that by proposition 3.16 with any $K \in \text{Gr}(3, V)$ we associate a linear section $Y_K = \Phi_{\text{Bl}} \mathcal{O}_{\mathbb{P}(K)}$ of Y .

Lemma 3.24. *The following facts about $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ hold:*

- (a) $\text{ch}(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))) = H + 7L/2 - P/6$.
- (b) $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ is the pushforward of the sheaf $R p_{Y_K*} p_K^* \mathcal{O}_{\mathbb{P}(K)}(2h)$ from Y_K .
- (c) $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ has no associated points of codimension 3.
- (d) $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ has associated points in codimension 2 if and only if $K \in \kappa(\mathbb{P}(A))$.
- (e) If $K \notin \kappa(\mathbb{P}(A))$ in $\text{Gr}(3, V)$, then $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ is torsion free on its support Y_K .

Proof. To check associated primes we check

$$\text{Tor}_p(\mathcal{O}_Z, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))$$

for $p = \text{codim}(Z)$ and Z any irreducible closed subset of Y .

- (a) Use additivity of the Chern character and formulas (28).

- (b) For our claim to hold, it is enough to check that the base change map for the leftmost square of diagram (83) is an isomorphism. This follows from lemma 1.33 as $\mathbb{P}_Y(\mathcal{U})$, $\mathbb{P}(K)$ and $\mathbb{P}(V)$ are smooth and S_K has expected dimension.
- (c) Any $\mathrm{Tor}_3(\Phi_{\mathrm{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)), -)$ vanishes as resolution (87) has length 2.
- (d) Each map from \mathcal{O}_Y to \mathcal{U}^* vanishes on a conic, so the first map in (87) vanishes on the intersection of a pencil of conics, which has codimension 2 if and only if all conics in the pencil are singular and share a component. By lemma 3.2 this implies that there is Tor_2 only with \mathcal{O}_L and if and only if $K \in \kappa(\mathbb{P}(A))$.
- (e) The computation of $\mathrm{ch}(\Phi_{\mathrm{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))$ shows that $\mathrm{Supp}(\Phi_{\mathrm{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))) = Y_S$. Moreover, as $K \notin \kappa(\mathbb{P}(A))$, all associated primes have codimension 1, so that the sheaf $\Phi_{\mathrm{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ is torsion free on its support by definition.

□

The following lemma describes further the correspondence between properties of $\mathbb{P}(K)$ in $\mathbb{P}(V)$ and properties of its transform $\Phi_{\mathrm{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$.

Lemma 3.25. *For a plane $\mathbb{P}(K) \subset \mathbb{P}(V)$ and a point $P \in Y$ the following conditions are equivalent*

- (a) $\mathrm{Tor}_2(\mathcal{O}_P, \Phi_{\mathrm{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2))) \neq 0$.
- (b) The line $p_Y^{-1}(P)$ is contained in S_K .
- (c) The line $p_V p_Y^{-1}(P)$ is contained in $\mathbb{P}(K)$.
- (d) All the conics in the pencil corresponding to $K \in \mathrm{Gr}(3, V) \cong \mathrm{Gr}(2, V^*)$ contain P .

Proof. Clearly, condition (b) is equivalent to condition (c). We will show that condition (a) is equivalent to conditions (c) and (d).

By adjunction formula

$$\mathcal{O}_P \otimes R p_{Y*} p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h) \cong R p_{Y*} \left(\mathcal{O}_{p_Y^{-1}(P)} \otimes p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h) \right)$$

The cohomology in degree -2 of the left hand side can be computed via the Grothendieck spectral sequence, so that

$$\mathrm{Tor}_2(\mathcal{O}_P, \Phi_{\mathrm{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))) = 0$$

if and only if

$$\mathrm{Tor}_2(\mathcal{O}_{p_Y^{-1}(P)}, p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h)) \tag{89}$$

has sections. The sheaf (89) can be computed via the Koszul complex for $\mathbb{P}(K)$. The result is the kernel of

$$\mathcal{O}_{p_Y^{-1}(P)} \rightarrow (V/K)^* \otimes \mathcal{O}_{p_Y^{-1}(P)}(1)$$

which can be either 0 or $\mathcal{O}_{p_Y^{-1}(P)}$. The second case occurs if and only if

$$p_V p_Y^{-1}(P) \notin \mathbb{P}(K)$$

so that we have proved the equivalence of conditions (a) and (c).

As for the equivalence of conditions (a) and (d), note that in order to compute

$$\mathrm{Tor}_2(\mathcal{O}_P, \Phi_{\mathrm{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2)))$$

it is enough to restrict the locally free resolution (88) to P , so that condition (a) is equivalent to the fact that

$$0 \rightarrow \mathcal{O}_Y \rightarrow (V/K)^* \otimes \mathcal{U}^*$$

is 0 at P , i.e. that P belongs to all conics in the pencil given by $(V/K)^* \subset V^*$.

The last part of the statement follows by letting P vary in Y and by corollary 3.12. \square

In the description of the moduli space of instantons with $c_2 = 3$ we will need the right adjoint $\Phi_{\mathrm{Bl}}^!$ to Φ_{Bl} . We discuss it in the following lemma.

Lemma 3.26. *Let $\Phi_{\mathrm{Bl}}^!$ be the right adjoint to Φ_{Bl} . Then the following facts hold.*

1. $\Phi_{\mathrm{Bl}}^!(-) = Rp_{V*}(Lp_Y^*(-) \otimes \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)[1])$
2. $\Phi_{\mathrm{Bl}}^!(\mathcal{O}_Y) \cong I_{\sigma(A)}(1)[1]$
3. *There is an exact sequence*

$$0 \rightarrow I_{\sigma(A)}(1) \rightarrow \Phi_{\mathrm{Bl}}^!(\mathcal{U})[-1] \rightarrow \mathcal{O}_{\mathbb{P}(V)}(-h) \rightarrow 0$$

Proof. 1. Since $\Phi_{\mathrm{Bl}} = Rp_{Y*} \circ Lp_Y^*$, its right adjoint $\Phi_{\mathrm{Bl}}^!$ is $Rp_{V*}(Lp_Y^*(-) \otimes \omega_{p_Y})[1]$. We have already computed ω_{p_Y} in 3.10, so by substitution we find

$$\Phi_{\mathrm{Bl}}^!(-) = Rp_{V*}(Lp_Y^*(-) \otimes \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)[1])$$

2. The sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E) \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h) \rightarrow \mathcal{O}_E(h) \rightarrow 0$$

is exact. Pushing it forward to $\mathbb{P}(V)$ we find

$$\Phi_{\mathrm{Bl}}^!(\mathcal{O}_Y) = Rp_{V*}\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)[1] = I_{\sigma(A)}(1)[1]$$

3. Recall that by proposition 3.10 on $\mathbb{P}_Y(\mathcal{U})$ there is a tautological extension

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-h) \rightarrow p_Y^*\mathcal{U} \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-2h + E) \rightarrow 0$$

Twisting it by $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)$ and pushing it forward we find that $\Phi_{\mathrm{Bl}}^!(\mathcal{U})$ is concentrated in degree 1 and that it sits in an exact sequence

$$0 \rightarrow I_{\sigma(A)} \rightarrow \Phi_{\mathrm{Bl}}^!(\mathcal{U})[-1] \rightarrow \mathcal{O}_{\mathbb{P}(V)}(-h) \rightarrow 0 \tag{90}$$

\square

Recall from definition 3.14 that we denote by S_K the total preimage of $\mathbb{P}(K)$ via p_V , and that S_K fits in diagram (83).

Lemma 3.27. *For any $K \in \text{Gr}(3, V)$*

$$Rp_{Y_K*}\mathcal{O}_{S_K} \cong \mathcal{O}_{Y_K}$$

Proof. As $Lp_{Y_K}^*\mathcal{O}_{Y_K} \cong \mathcal{O}_{S_K}$, we only need to compute $Rp_{Y_K*}\mathcal{O}_{S_K}$. Note that in 3.16 we have checked that $i_{Y_K*}Rp_{Y_K*}\mathcal{O}_{S_K}$ and \mathcal{O}_{Y_K} are isomorphic, so that by exactness of i_{Y_K*} we find that $Rp_{Y_K*}\mathcal{O}_{S_K}$ is equivalent to a complex with cohomology concentrated in degree 0 and isomorphic to \mathcal{O}_{Y_K} . \square

Lemma 3.28. *For $K \notin D_{tri}$, the map $p_{Y_K} : S_K \rightarrow Y_K$ is an isomorphism.*

Proof. First, we prove that p_{Y_K} has relative dimension 0. Clearly, the only irreducible curves which p_Y contracts are the fibers of points of Y . By lemma 3.12 such curves are the strict transforms of trisecants to $\sigma(A)$ inside $\mathbb{P}(V)$. As $K \notin D_{tri}$, $\mathbb{P}(K)$ does not contain trisecants, so that S_K does not contain the strict transform of any trisecant.

Finally note that p_{Y_K} is proper as p_Y is proper and that the canonical map

$$\mathcal{O}_{Y_K} \rightarrow p_{Y_K*}\mathcal{O}_{S_K}$$

is an isomorphism by 3.27. By lemma 1.35, p_{Y_K} is an isomorphism. \square

4 Instantons

In section 4.1 we introduce instantons on Fano 3-folds and state the main results that are already known about them. Section 4.2 is devoted to instantons on Y and does not contain new results, but it contains a detailed discussion about the description of \mathcal{MI}_n as a geometric quotient of an affine variety. Section 4.3 contains the main new results about the splitting type of instantons on lines and conics.

4.1 Instantons on Fano 3-folds

Let X be a smooth projective variety of dimension $\dim X$. One says that X is Fano when its anti-canonical class $-\omega_X$ is ample. Assume that $\text{Pic}(X) = \mathbb{Z}$ and denote the ample generator by $\mathcal{O}_X(1)$. The index ι_X of X is the maximal integer dividing the canonical class ω_X .

By [KO73], we have $\iota_X \leq \dim X + 1$. Moreover, the only Fano variety of index $\dim X + 1$ is the projective space and the only Fano variety of index $\dim X$ is the non-singular quadric hypersurface inside the projective space of dimension $\dim X + 1$. In dimension 3 the classification is complete [Isk80] [MU83]. A list of the deformation classes of Fano 3-folds can be found for example in [IP99].

In analogy with \mathbb{P}^3 , where the twist by $\mathcal{O}(-2)$ is the twist by a square root of the canonical class, Kuznetsov gives the following definition [Kuz12] of instanton on a Fano 3-fold of index 2.

Definition 4.1. An instanton E on a Fano 3-fold X of index 2 is a rank 2 μ -stable vector bundle such that $c_1(E) = 0$ and $H^1(E(-1)) = 0$. The second Chern class $c_2(E) = n$ is called the charge of E .

Remark 4.2. One can easily check that

$$\mathrm{ch}(E) = 2 - c_2(E) = 2 - n \cdot L \quad (91)$$

Moreover, it follows from the stability (and from the anti-selfduality) of E that

$$h^1(E) = c_2(E) - 2 = n - 2$$

and that $h^2(E) = h^3(E) = 0$. It follows that there are no instantons with $c_2(E) < 2$.

In the more general setting of a Fano 3-fold of arbitrary index, Faenzi gives a definition of instanton bundle in [Fae11].

Definition 4.3. An instanton E on a Fano 3-fold X is the normalization of a rank 2 Gieseker-stable vector bundle F such that $F \cong F^* \otimes \omega_X$ and $H^1(F) = 0$.

In the next sections we will work on the Fano 3-fold Y defined in section 2, but in this section we will recall a few results that are already known about instantons on Fano 3-folds, starting from the following remark.

Remark 4.4. For Fano 3-folds of index 2 the definitions 4.1 and 4.3 are equivalent. This follows directly from lemma 1.23.

We will denote by $\mathcal{MI}_{X,k}$ the moduli space of instantons of charge k on X . When working with $X = Y_5 = Y$ we omit the X in the index, so that \mathcal{MI}_k stands for $\mathcal{MI}_{Y,k}$. There are some simple constraints on the values of k and of some discrete parameters of X which guarantee non-emptiness of $\mathcal{MI}_{X,k}$ and generic smoothness of at least an irreducible component of it.

Theorem 4.5 ([Fae11]). *The emptiness of $\mathcal{MI}_{X,k}$ depends only on k and on discrete invariants of X . If it is non-empty, $\mathcal{MI}_{X,k}$ has a generically smooth irreducible component of dimension depending only on k and on discrete invariants of X .*

A classical result about instantons on \mathbb{P}^3 is the existence of a monadic description for them. More precisely, for any instanton E of charge k on \mathbb{P}^3 there is an anti-selfdual complex (called monad)

$$\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus n} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\oplus 2n+2} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus n}$$

such that the first map is injective, the last is surjective and the middle cohomology is E . From this point of view, the instantonic condition $H^1(E(-2)) = 0$ and the stability of E provide the orthogonality of E to $\mathcal{O}_{\mathbb{P}^3}(2)$, and the monad is the decomposition of E with respect to the selfdual semi-orthogonal decomposition for $\mathcal{O}_{\mathbb{P}^3}(2)^\perp$ inside $D^b(\mathbb{P}^3)$.

This point of view generalizes to all X Fano threefolds of Picard number 1 having a full exceptional collection. Such X are completely classified and they are:

- \mathbb{P}^3 ;
- Q , a smooth quadric inside \mathbb{P}^4 ;
- $Y \subset \text{Gr}(2, 5)$, the intersection of 3 linear sections of $\text{Gr}(2, 5)$
- $X_{22} \subset \text{Gr}(3, 7)$, the zero locus of three sections of $\Lambda^2 \mathcal{U}_3^*$, where \mathcal{U}_3 is the rank 3 tautological bundle.

For any X in the above list and for any integer k , one can imitate the monadic description on \mathbb{P}^3 . The identification of the moduli spaces of instantons on the above Fano threefolds with geometric quotients of spaces of linear forms is carried out in [Fae11, thm. B].

In all other cases, only a partial decomposition is available for $D^b(X)$. When the index of X is 2, the derived category of X decomposes as

$$D^b(X) = \langle \mathcal{B}_X, \mathcal{O}_X, \mathcal{O}_X(1) \rangle$$

where \mathcal{B}_X is called the non-trivial part of $D^b(X)$.

The instantonic condition $H^1(E(-1))$ and the stability of E imply the orthogonality of E with respect to $\mathcal{O}_X(1)$. The projection \tilde{E} of an instanton E to \mathcal{B}_X sits in an exact sequence

$$0 \rightarrow E \rightarrow \tilde{E} \rightarrow H^1(E) \otimes \mathcal{O}_X \rightarrow 0 \quad (92)$$

as the universal extension of \mathcal{O}_X by E . Conversely, given \tilde{E} with the appropriate Chern character and H^0 , if \tilde{E} is self dual with respect to a certain anti-autoequivalence of $(\mathcal{O}_X, \mathcal{O}_X(1))^\perp$, it is proved in [Kuz12, thm 3.10] that one can reconstruct uniquely an instanton E such that \tilde{E} is its universal extension.

4.2 Instantons on Y

In the case of instantons on Y , one can write an anti-selfdual monad in terms of the exceptional collection $\mathcal{U}, \mathcal{O}_Y, \mathcal{U}^*, \mathcal{O}_Y(1)$. It is often convenient to work with the decomposition of an instanton with respect to collection (27), that is to say $\mathcal{U}, \mathcal{U}^\perp, \mathcal{O}_Y, \mathcal{O}_Y(1)$.

Fix a positive integer n and an n -dimensional vector space H . Recall that A is both the space of linear sections which cut Y inside $\text{Gr}(2, V)$ and the space of maps from \mathcal{U} to \mathcal{U}^\perp by lemma 2.14. As a consequence, given $\gamma \in A \otimes S^2 H^*$ one can associate to it a map γ' from $H \otimes \mathcal{U}$ to $H^* \otimes \mathcal{U}^\perp$ and a map $\hat{\gamma}$ as in the following definition.

Definition 4.6. Given $\gamma \in A \otimes S^2 H^*$, we denote by γ' the composition

$$\gamma' : H \otimes \mathcal{U} \xrightarrow{\gamma} A \otimes H^* \otimes \mathcal{U} \xrightarrow{H^* \otimes \text{ev}} H^* \otimes \mathcal{U}^\perp$$

and by $\hat{\gamma}$ the map

$$\hat{\gamma} : H \otimes V \rightarrow H^* \otimes V^*$$

obtained by polarizing γ with respect to V .

Remark 4.7. Note that $\hat{\gamma} = \text{Hom}(\gamma', \mathcal{O}_Y)$ under the canonical isomorphisms $\text{Hom}(\mathcal{U}^\perp, \mathcal{O}_Y) \cong V$ and $\text{Hom}(\mathcal{U}, \mathcal{O}_Y) \cong V^*$. Moreover, we will often write

$$\gamma : A \rightarrow S^2 H^*$$

by means of the canonical SL_2 -invariant identification between A^* and A .

It turns out ([Kuz12], [Fae11]) that every instanton E on Y is isomorphic to the middle cohomology of a 3-term complex

$$0 \rightarrow H \otimes \mathcal{U} \rightarrow H^* \otimes \mathcal{U}^\perp \rightarrow H' \otimes \mathcal{O}_Y \rightarrow 0 \quad (93)$$

with $\dim(H') = \dim(H) - 2 = n - 2$ and such that the first map is fiberwise injective and the last is surjective. Complex (93) is usually called a *monad* for E . The following theorem is an improvement of this description.

Theorem 4.8 ([Fae11], [Kuz12]). *Let H^* be a vector space of dimension n and let $G := GL(H)/\{\pm 1\}$. Denote by $M_n(Y)$ the set of all $\gamma \in A \otimes S^2 H^*$ which satisfy the following conditions*

- (i) *the map $\gamma' : H \otimes \mathcal{U} \rightarrow H^* \otimes \mathcal{U}^\perp$ is a fiberwise monomorphism of vector bundles,*
- (ii) *the rank of the map $\hat{\gamma} : H \otimes V \rightarrow H^* \otimes V^*$ equals $4n + 2$.*

Then the coarse moduli space $\mathcal{MI}_n(Y)$ of instantons of charge n on Y is the GIT-quotient $M_n(Y)/G$.

Remark 4.9 (Scheme structure). More precisely, $M_n := M_n(Y)$ should be given the following scheme structure. Consider the polarized tautological map

$$\mathcal{O}_{A \otimes S^2 H^*} \otimes H \otimes V \rightarrow \mathcal{O}_{A \otimes S^2 H^*} \otimes H^* \otimes V^* \quad (94)$$

on $A \otimes S^2 H^*$. Take the degeneration locus of rank $4n + 2$, i.e. the locus where the map (94) has rank exactly $4n + 2$. This is a locally closed subscheme in $A \otimes S^2 H^*$.

Next, consider over $A \otimes S^2 H^* \times Y$ the universal composition

$$\mathcal{O}_{A \otimes S^2 H^*} \otimes H \boxtimes \mathcal{U} \rightarrow \mathcal{O}_{A \otimes S^2 H^*} \otimes H^* \boxtimes A \otimes \mathcal{U} \rightarrow \mathcal{O}_{A \otimes S^2 H^*} \otimes H^* \boxtimes \mathcal{U}^\perp \quad (95)$$

It is a map of vector bundles of rank $2n$ and $3n$ and it is fiberwise a monomorphism out of a closed subset. As Y is proper, the image of this subset in $A \otimes S^2 H^*$ is again closed, so that the condition that the map (95) is injective at all points of Y defines an open subscheme of $A \otimes S^2 H^*$.

The intersection of the two subschemes which we have just defined, provides M_n with a natural scheme structure, which is the one used in theorem 4.8.

Remark 4.10 (Geometric quotient). Even though theorem 4.5 says that for each $n \geq 2$ there is a generically smooth irreducible component of \mathcal{MI}_n , nothing is known about possible other components when $n \geq 4$. It is not even known whether in general M_n is reduced or not.

On the other end, in the case $n = 2$ or $n = 3$, condition 4.8 (ii) is open. It follows that M_2 and M_3 are smooth.

For arbitrary n , it is straightforward to check that the orbits of G in M_n are closed and that all stabilizers are trivial (i.e. that all points of M_n are GIT-stable with respect to the action of G). It suffices to note that if an orbit is not closed, then there are two non-isomorphic monads mapping to the same point in the coarse moduli space of instantons, which is impossible by the naturality of the Beilinson spectral sequence. It follows that \mathcal{MI}_n is a geometric quotient of M_n by G , so that \mathcal{MI}_2 and \mathcal{MI}_3 are smooth.

Remark 4.11 (Coarse moduli space). Denote by \mathfrak{MI}_n the functor associating with a scheme S the set of families of instantons on Y up to twists by line bundles on S . In order to check that M_n/G is the coarse moduli space for \mathfrak{MI}_n one has to check a few facts, namely that there is a natural transformation from \mathfrak{MI}_n to M_n/G which is bijective on closed points and that it corepresents \mathfrak{MI}_n .

The key point in the proof of the above facts is that over $M_n \times Y$ there is a universal monad, as we are about to see. Note that over $A \otimes S^2 H^*$ there is a map

$$0 \rightarrow H \boxtimes \mathcal{U} \rightarrow H^* \boxtimes \mathcal{U}^\perp \rightarrow 0 \quad (96)$$

called the universal short monad.

By property 4.8 (i), the cohomology of the universal short monad (96) on $M_n \subset A \otimes S^2 H^*$ is locally free of rank n .

By property 4.8 (ii) over M_n there is a universal complex

$$H \boxtimes \mathcal{U} \rightarrow H^* \boxtimes \mathcal{U}^\perp \rightarrow \mathcal{D}^* \boxtimes \mathcal{O}_Y \quad (97)$$

where \mathcal{D} is a vector bundle of rank $n - 2$ sitting in an exact anti-selfdual exact sequence

$$0 \rightarrow \mathcal{D} \rightarrow H \otimes V \otimes \mathcal{O}_{A \otimes S^2 H^*} \rightarrow H^* \otimes V^* \otimes \mathcal{O}_{A \otimes S^2 H^*} \rightarrow \mathcal{D}^* \rightarrow 0 \quad (98)$$

We will call the object in (97) the universal monad, as its restriction to each point of $A \otimes S^2 H^*$ is isomorphic to

$$0 \rightarrow H \otimes \mathcal{U} \rightarrow H^* \otimes \mathcal{U}^\perp \rightarrow H' \otimes \mathcal{O}_Y \rightarrow 0 \quad (99)$$

The universal monad (97) is $GL(H)$ -equivariant but not G -equivariant, as -1 acts non-trivially on it. When n is odd, one can twist the $GL(H)$ action by the character \det and make it G -equivariant. When n is even all characters of $GL(H)$ are trivial on -1 , so that one cannot use the same method. We will discuss the behaviour for $n = 2$ in proposition 5.12.

In order to prove that M_n/G is a coarse moduli space, one should first provide a natural transformation from the functor \mathfrak{MI}_n to M_n/G . This can be done in the same way as in theorem [OSS80, 4.1.12]. Given a family of instantons parametrized by S , use the Beilinson spectral sequence to get a monad for it, restrict to open subsets S_i where the bundles in the monad trivialize to get maps from S_i to M_n and finally show that the induced maps to M_n/G are compatible and therefore glue to a map from S to M_n/G .

The proof that this natural transformation is bijective on closed points can be carried out exactly in the same way of [OSS80, 4.1.12].

As we do not know whether M_n is reduced or not, we have to be a bit more careful while proving that M_n/G corepresents $\mathfrak{M}\mathfrak{J}_n$. The proof in [OSS80, 4.1.12] uses the fact that the space of monads is reduced, and we want to avoid using it. The main problem is in the construction of a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}\mathfrak{J}_n & \xrightarrow{\Psi} & \mathrm{Hom}(-, N) \\ \uparrow \scriptstyle E & & \uparrow \scriptstyle \vdots \\ \mathrm{Hom}(-, M_n) & \xrightarrow{\pi} & \mathrm{Hom}(-, M_n/G) \end{array}$$

once a natural transformation Ψ is given. Here E is the map from M_n to $\mathfrak{M}\mathfrak{J}_n$ corresponding to the cohomology of the universal monad (97). Note that it is not clear that the map $\Psi \circ E$ descends to a map from M_n/G , as the universal E is not necessarily G -equivariant, but only $GL(H)$ -equivariant.

On the other hand, if we want to check that the diagram

$$\begin{array}{ccc} G \times M_n & \longrightarrow & M_n \\ \downarrow & & \downarrow \\ M_n & \longrightarrow & N \end{array} \tag{100}$$

is commutative, we can replace $G \times M_n$ by a flat cover. As a flat cover we choose $GL(H) \times M_n$, so that by $GL(H)$ -equivariance of E we can deduce the commutativity of

$$\begin{array}{ccc} GL(H) \times M_n & \longrightarrow & M_n \\ \downarrow & & \downarrow \\ M_n & \longrightarrow & N \end{array}$$

and therefore of (100).

In the above way one can avoid referring to closed points and complete the proof of the fact that M_n/G is a coarse moduli space following the lines of [OSS80, 4.1.12].

4.3 Jumping lines and conics

We are now going to discuss the restriction of E to rational curves in Y . By rational curve C we mean a scheme of pure dimension 1 with $h^1(\mathcal{O}_C) = 0$ and $h^0(\mathcal{O}_C) = 1$. Let us first recall the following definition.

Definition 4.12. A rational curve C is jumping for an instanton E if $E|_C$ is not trivial. If C is smooth we will say that C is n -jumping for E if $E|_C \cong \mathcal{O}_C(-n) \oplus \mathcal{O}_C(n)$. In this case we will call n the *order of jump* of E at C .

Recall that in proposition 2.20 and 2.32 we have constructed the universal families of lines

$$\begin{array}{ccc} & \mathcal{L} & \\ r_Y \swarrow & & \searrow r_A \\ Y & & \mathbb{P}(A) \end{array} \quad (101)$$

and of conics

$$\begin{array}{ccc} & \mathcal{C} & \\ q_Y \swarrow & & \searrow q_V \\ Y & & \mathbb{P}(V^*) \end{array} \quad (102)$$

Definition 4.13. The Fourier–Mukai transforms from $D^b(Y)$ to $D^b(\mathbb{P}(V^*))$ with kernel $\mathcal{O}_{\mathcal{L}}$ and $\mathcal{O}_{\mathcal{C}}$ are denoted by

$$\Phi_{\mathcal{L}} := Rr_{A*} Lr_Y^* : D^b(Y) \rightarrow D^b(\mathbb{P}(A)) \quad (103)$$

$$\Phi_{\mathcal{C}} := Rq_{V*} Lq_Y^* : D^b(Y) \rightarrow D^b(\mathbb{P}(V^*)) \quad (104)$$

Theorem 4.14 and corollary 4.16 clarify the relation between jumping lines and the Fourier–Mukai transform $\Phi_{\mathcal{L}}(E(-1))$.

Theorem 4.14 ([Kuz12]). *Let E be an instanton on Y and*

$$\gamma_E \in A \otimes S^2 H^* \cong A^* \otimes S^2 H^*$$

via the canonical identification of A and A^ . Then there is a distinguished triangle*

$$\Phi_{\mathcal{L}}(E(-1)) \rightarrow H \otimes \mathcal{O}_{\mathbb{P}(A)}(-3) \xrightarrow{\gamma_E} H^* \otimes \mathcal{O}_{\mathbb{P}(A)}(-2)$$

We include a sketch of a proof of theorem 4.14 different from the one which is given in [Kuz12].

Proof. First, as $\mathcal{O}_L(-1)$ is acyclic,

$$\Phi_{\mathcal{L}}(E(-1)) = \Phi_{\mathcal{L}}(\tilde{E}(-1))$$

where \tilde{E} is the acyclic extension of E defined by (92). By theorem 4.8 we have a resolution

$$0 \rightarrow H \otimes \mathcal{U} \rightarrow H^* \otimes \mathcal{U}^\perp \rightarrow \tilde{E} \rightarrow 0$$

By lemma 2.17, $\Phi_{\mathcal{L}}(\mathcal{U}(-1))$ and $\Phi_{\mathcal{L}}(\mathcal{U}^\perp(-1))$ are line bundles over $\mathbb{P}(A)$ shifted by -1 . By means of the resolution for $\mathcal{O}_{\mathcal{L}}$ given in proposition 2.20 one can check that they are $\mathcal{O}_{\mathbb{P}(A)}(-3)$ and $\mathcal{O}_{\mathbb{P}(A)}(-2)$. Finally, by computing $\Phi_{\mathcal{L}}(\mathcal{O}_L(-1))$ one can also check that

$$\Phi_{\mathcal{L}} : \text{Hom}(\mathcal{U}(-1), \mathcal{U}^\perp(-1)) \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}(A)}(-3), \mathcal{O}_{\mathbb{P}(A)}(-2))$$

is an isomorphism. □

We introduce a shorthand notation for $\Phi_{\mathcal{L}}(E(-1))[1]$.

Definition 4.15. We will call $\Phi_{\mathcal{L}}(E(-1))[1]$ the *object of jumping lines of E* and we will denote it by JL_E . If the generic line is not jumping for E , then by theorem 4.14 it is equivalent to a sheaf and we will therefore call it the *sheaf of jumping lines of E* .

Corollary 4.16. *Let $L \subset Y$ be a line, then $E|_L = (-n, n)$ if and only if the corank of γ_E at a_L is n .*

Proof. The integer n such that $E|_L = (-n, n)$ is the dimension of $H^1(E(-1))$. By flat base change at the point representing L for the diagram

$$\begin{array}{ccc} L & \longrightarrow & \mathcal{L} \\ \downarrow & & \downarrow \\ a_L & \longrightarrow & \mathbb{P}(A) \end{array}$$

we have that the derived restriction of $\Phi_{\mathcal{L}}(E(-1))$ at a_L is the cohomology $H^\bullet(E(-1))$. By theorem 4.14 the dimension of $H^1(E(-1))$ is the corank of γ_E at a_L . \square

In the following theorem we will prove that for an instanton E of charge n the order of jump at a line is strictly less than n . This theorem is a key step in the description 5.8 of instantons of charge 2. It is also useful in higher charge as it shows that the map

$$\gamma : A^* \rightarrow S^2 H^*$$

associated with a monad for an instanton E induces a regular map from $\mathbb{P}(A^*)$ to $\mathbb{P}(S^2 H^*)$.

Theorem 4.17. *Let E be an instanton and*

$$\gamma' : H \otimes \mathcal{U} \rightarrow H^* \otimes \mathcal{U}^\perp \tag{105}$$

the initial part of a monad for E . Let $\gamma : A^ \rightarrow S^2 H^*$ the map associated to the monad. Then γ is injective.*

Proof. Assume there is a such that $\gamma(a) = 0$. By base change $\gamma(a)$ is H^1 of

$$\gamma'|_{L_a} : H \otimes \mathcal{U}|_L \rightarrow H^* \otimes \mathcal{U}^\perp|_L$$

which by lemma 2.17 is isomorphic to

$$\gamma'|_{L_a}(-1) : H \otimes (\mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)) \xrightarrow{0} H^* \otimes (\mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2))$$

As γ' represents an instanton, by theorem 4.8 it is fiberwise injective, so that it induces a fiberwise injection

$$H \otimes \mathcal{O}_L(-2) \xrightarrow{0} H^* \otimes (\mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2)) \tag{106}$$

which becomes 0 when we apply H^1 . It follows that the map (106) is a fiberwise injection factoring via $H^* \otimes \mathcal{O}_L(-1)$. As there is no fiberwise injection from $H \otimes \mathcal{O}_L(-2)$ to $H^* \otimes \mathcal{O}_L(-1)$, we conclude that it is impossible to have $\gamma(a) = 0$. \square

Corollary 4.18. *The order of jump of E at a line L is at most $c_2(E) - 1$.*

Proof. By corollary 4.16 the order of jump at a line L is the corank of $\gamma(a_L)$. By theorem 4.17 the rank of $\gamma(a_L)$ is at least 1, so that the corank is at most $c_2(E) - 1$. \square

It is not known whether \mathbf{JL}_E is supported on the whole $\mathbb{P}(A)$ or not (see e.g. [Kuz12, conj. 3.16]). The reason is that the usual Grauert–Mülich theorems, such as theorem 1.26, can only be used provided the family of lines through a general point is irreducible. By corollary 2.24 this is clearly not the case for Y .

If we turn our attention to conics, the situation changes completely: by corollary 2.41 the family of conics passing through a point of Y is parametrized by \mathbb{P}^2 , so that the assumptions of theorem 1.26 are satisfied.

Recall that by definition 4.12 a conic is jumping for E if and only if $E|_C$ is not trivial. We are now going to construct an analogue of \mathbf{JL}_E in the case of conics.

Remark 4.19. If L is a line, then the restriction $E(-1)|_L$ is acyclic if and only if L is not jumping for E . Analogously, by corollary 2.40, if C is a smooth conic, then the restriction $E \otimes \mathcal{U}|_C$ is acyclic if and only if C is not jumping for E .

The above remark suggests that a possible analogue of \mathbf{JL}_E in the case of conics could be a shift of $\Phi_C(E \otimes \mathcal{U})$. We will now check that the restriction of $E \otimes \mathcal{U}$ to singular conics detects whether C is jumping or not for E , that is to say whether $E|_C$ is trivial or not.

Recall that by conic we mean a subscheme C of Y with Hilbert polynomial $2t + 1$ and that this implies that C has pure dimension 1 and $h^1(\mathcal{O}_C) = 0$ (see remark 2.31). Let C be a singular conic and denote by L_1, L_2 its components (in the case C is a double line, take $L_1 = L_2$). Then, lemma 4.21 and proposition 4.22 provide us with a cohomological way to check whether a conic is jumping or not.

As some conics in Y are non-reduced, we will need the following remark.

Remark 4.20. For each singular conic C (possibly non-reduced) the ideal \mathcal{I}_1 of L_1 in C is isomorphic to $\mathcal{O}_{L_2}(-1)$. The reason is that by additivity the Hilbert polynomial of \mathcal{I}_1 is t , which shows that it is schematically supported on a component of C . Moreover it has no associated points of dimension 0 because it is a subsheaf of \mathcal{O}_C . This is enough to conclude that it is isomorphic to $\mathcal{O}_{L_2}(-1)$, so that there is an exact sequence

$$0 \rightarrow \mathcal{O}_{L_2}(-1) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{L_1} \rightarrow 0 \quad (107)$$

both in the case $L_1 \neq L_2$ and in the case $L_1 = L_2$

Lemma 4.21. *Let F be a vector bundle of rank r on a singular conic C with (possibly coinciding) components L_1, L_2 . If F is trivial on L_1 and L_2 then F is trivial on C .*

Proof. As F is trivial on L_1 , we can choose an isomorphism

$$s_1 : \mathcal{O}_{L_1}^{\oplus r} \rightarrow F|_{L_1}$$

Tensor the exact sequence (107) by F . As F is trivial on L_2 , the restriction $F(-1)|_{L_2}$ is acyclic, so that s_1 lifts to

$$s : \mathcal{O}_C^{\oplus r} \rightarrow F$$

As $\mathcal{O}_C^{\oplus r}$ and F are vector bundles of the same rank, in order to check that s is an isomorphism it is enough to check that s is surjective. This is equivalent to surjectivity of s_1 and $s_2 = s|_{L_2}$. As F is trivial on each L_i and L_i is irreducible, it is enough to check that each s_i is surjective at some closed reduced point $P_i \in L_i$. Choose $P_1 = P_2 = P \in L_1 \cap L_2$, then

$$s_1|_P \cong s_2|_P$$

and the first one is an isomorphism by definition. \square

Proposition 4.22. *A conic C is jumping for an instanton E if and only if*

$$H^\bullet(E \otimes \mathcal{U}|_C) \neq 0$$

Proof. Assume $E|_C$ is trivial on C . Then there is an isomorphism between $E \otimes \mathcal{U}|_C$ and $\mathcal{U}|_C^{\oplus 2}$. By lemma 2.38 we have $H^\bullet(E \otimes \mathcal{U}|_C) = 0$.

In the other direction, for C smooth conic, $H^\bullet(E \otimes \mathcal{U}|_C) = 0$ implies $E|_C$ trivial by 2.40. For C singular conic, by lemma 4.21 it is enough to prove that $H^\bullet(E \otimes \mathcal{U}|_C) = 0$ implies $E|_L$ trivial for L component of C . Tensoring sequence (107) by $E \otimes \mathcal{U}$ we find is an injection

$$H^0(E \otimes \mathcal{U} \otimes \mathcal{O}_L(-1)) \rightarrow H^0(E \otimes \mathcal{U}|_C) = 0$$

so that also $H^0(E \otimes \mathcal{U} \otimes \mathcal{O}_L(-1))$ vanishes. By lemma 2.17 and $c_1(E) = 0$ we deduce that $E|_L$ is trivial. \square

The following remark is analogous to remark 4.19, but concerns also singular conics.

Remark 4.23. By proposition 4.22 a conic C is jumping for E if and only if, in the notation of definition 2.34, $v_C \in \mathbb{P}(V^*)$ lies in the support of $\Phi_C(E \otimes \mathcal{U})$.

While the family of lines through a point in Y is disconnected, we have proved in 2.41 that the family of conics through a point P in Y is an irreducible variety. This allows us to use 1.26 in order to state and prove the following theorem.

Theorem 4.24. *For an instanton E on Y , the generic conic is not jumping. Moreover, $\Phi_C(E \otimes \mathcal{U})[1]$ is the cokernel of an injection*

$$H \otimes (\mathcal{O}(-2) \oplus (A \otimes \mathcal{O}(-1))) \rightarrow H^* \otimes ((A^* \otimes \mathcal{O}(-1)) \oplus \mathcal{O}) \quad (108)$$

and is equivalent to a sheaf supported on a divisor.

Proof. We will use theorem 1.26 on the family of smooth conics in Y . The flatness condition (1) holds as the projection from \mathcal{C} to Y is the projection from $\mathbb{P}_Y(\mathcal{U}^\perp)$ to Y by proposition 2.32. The condition (2) about the irreducibility of the generic fiber follows from corollary 2.41. The proportionality condition (3) is always true on Y as each even cohomology group is isomorphic to \mathbb{Z} .

We are now interested in the splitting type of an instanton E on the generic smooth conic. By theorem 1.26 we have that if $\mathrm{ST}_C(E) = (-j, j)$ with $j \geq 0$, then

$$2j \leq -\mu_{\min}(\mathcal{T}_{C/Y}/(\mathcal{C}/\mathbb{P}(V^*))) \quad (109)$$

where the right hand side is the minimal slope in the Harder-Narasimhan filtration of the relative tangent bundle restricted to the general fiber of $\mathcal{C} \rightarrow \mathbb{P}(V^*)$.

This minimal slope can be computed as follows. First, by proposition 2.32, identify the universal conic (together with its two projections to Y and $\mathbb{P}(V^*)$) with $\mathbb{P}_Y(\mathcal{U}^\perp)$.

It follows that there is a relative Euler exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{U}^\perp \otimes p_V^* \mathcal{O}_{\mathbb{P}(V^*)}(1) \rightarrow \mathcal{T}_{C/Y} \rightarrow 0$$

which by corollary 2.40 restricts to any smooth conic C as

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C \oplus \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1) \rightarrow \mathcal{T}_{C/Y}|_C \rightarrow 0$$

This is enough to conclude that $\mathrm{ST}_C(\mathcal{T}_{C/Y}) = (-1, -1)$, so that inequality (109) becomes $2j \leq 1$, that is to say $j = 0$.

In order to show that $\Phi_{\mathcal{C}}(E \otimes \mathcal{U})$ is equivalent to a shifted sheaf, note that as $\Phi_{\mathcal{C}}(\mathcal{U}) = 0$ we have

$$\Phi_{\mathcal{C}}(E \otimes \mathcal{U}) \cong \Phi_{\mathcal{C}}(\tilde{E} \otimes \mathcal{U})$$

where \tilde{E} is the acyclic extension defined in (92). Moreover, for $\Phi_{\mathcal{C}}(\tilde{E} \otimes \mathcal{U})$ we have a distinguished triangle

$$H \otimes \Phi_{\mathcal{C}}(\mathcal{U} \otimes \mathcal{U}) \rightarrow H^* \otimes \Phi_{\mathcal{C}}(\mathcal{U}^\perp \otimes \mathcal{U}) \rightarrow \Phi_{\mathcal{C}}(E \otimes \mathcal{U})$$

Via Borel–Bott–Weil and the resolution for $\mathcal{O}_{\mathcal{C}}$ of proposition 2.32, the above distinguished triangle becomes

$$H \otimes (\mathcal{O}(-2) \oplus (A \otimes \mathcal{O}(-1))) \rightarrow H^* \otimes ((A^* \otimes \mathcal{O}(-1)) \oplus \mathcal{O}) \rightarrow \Phi_{\mathcal{C}}(E \otimes \mathcal{U})[1]$$

where we write \mathcal{O} instead of $\mathcal{O}_{\mathbb{P}(V^*)}$ for brevity. By the first part of this proof, $\Phi_{\mathcal{C}}(E \otimes \mathcal{U})$ vanishes at the generic point of $\mathbb{P}(V^*)$, so that the map

$$H \otimes (\mathcal{O}(-2) \oplus (A \otimes \mathcal{O}(-1))) \rightarrow H^* \otimes ((A^* \otimes \mathcal{O}(-1)) \oplus \mathcal{O}) \quad (110)$$

is injective at the generic point of $\mathbb{P}(V^*)$. By looking at the determinant of (110), it follows that $\Phi_{\mathcal{C}}(E \otimes \mathcal{U})[1]$ is equivalent to a sheaf supported on a divisor. \square

In analogy with definition 4.15, we give the following definition.

Definition 4.25. We will call $\Phi_{\mathcal{C}}(E \otimes \mathcal{U})[1]$ the *sheaf of jumping conics* of E and we will denote it by JC_E .

Remark 4.26. By corollary 4.24 the object JC_E is equivalent to a sheaf supported on a divisor. By remark 4.23 a point $v_C \in \mathbb{P}(V^*)$ is in the support of JC_E if and only if C is jumping for E .

The next corollary is a consequence of 4.22 and 4.24 and allows to translate statements about jumping lines into statements about jumping conics. Hopefully, this can be a step towards a proof of the following conjecture.

Conjecture 4.27. For an instanton E on Y the generic line is not jumping.

Corollary 4.28. For an instanton E the following conditions are equivalent.

1. The generic line is jumping for E .
2. JC_E contains the divisor of singular conics.

Proof. To prove that condition 2 implies condition 1 we can argue as follows. Assume that the generic line is not jumping for E , so that the support of JL_E is a curve. It follows that $\mathrm{JL}_E \times \mathrm{JL}_E$ is a surface, while by 2.26 the variety of intersecting lines IQ is 3-dimensional. As a consequence, there is at least a pair of intersecting lines L_1, L_2 such that E restricts trivially to both of them. By lemma 4.21, E is trivial on the conic which is the union of L_1 and L_2 , so that the support of JC_E not contain the locus of singular conics.

Condition 1 implies condition 2 by lemma 4.21. \square

Proposition 4.29. The divisor of jumping conics for an instanton E of charge n has degree n (counted with multiplicity).

Remark 4.30. By counted with multiplicity we mean the following. The support of JC_E can be reducible and have many components J_i : n will be the sum of $\deg(J_i) \cdot O_i$ where O_i is the order of jump of E at the generic curve of J_i .

If a component J_i is the divisor of reducible conics, then by corollary 4.28 the generic line is jumping for E . In this case O_i is $2k - 1$, where k the order of jump of E at the generic line.

Proof. By resolution (108), the leading term of the Chern character of JC_E is n times the class of a quadric in $\mathbb{P}(V^*)$. This shows that the sum

$$\sum_i \deg(J_i) \cdot \mathrm{rank}(\mathrm{JC}_E, J_i) = 2n$$

where $\mathrm{rank}(\mathrm{JC}_E, J_i)$ is the rank of JC_E at the generic point of J_i . Our claim will be that for any component J of JC_E we have

$$\mathrm{rank}(\mathrm{JC}_E, J_i) = 2 \cdot O_i$$

Note that by base change we always have the equality

$$\mathrm{rank}(\mathrm{JC}_E, J_i) = h^1(E \otimes \mathcal{U}|_C)$$

where C is a general conic in J_i .

Assume now that the generic conic C in J_i is smooth. Then by corollary 2.40 we have

$$\text{rank}(\text{JC}_E, J_i) = h^1(E \otimes \mathcal{U}|_C) = 2 \cdot h^1(E|_C) = 2 \cdot O_i$$

The only case which is left is that in which J_i is the divisor of reducible conics. Let C be a general reducible reduced conic with components L_1, L_2 . By lemma 4.31

$$h^0(E \otimes \mathcal{U}|_C) = 4k - 2$$

where k is the order of jump of E at L_i . Finally as $\chi(E \otimes \mathcal{U}|_C) = 0$ we deduce

$$\text{rank}(\text{JC}_E, J_i) = h^1(E \otimes \mathcal{U}|_C) = 2 \cdot O_i$$

□

Lemma 4.31. *If the generic line is k -jumping for E , $k > 0$, then $h^0(E \otimes \mathcal{U}|_C) = 4k - 2$.*

Proof. Denote the intersection point of L_1 and L_2 by P . Tensor the exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{L_1} \oplus \mathcal{O}_{L_2} \rightarrow \mathcal{O}_P \rightarrow 0$$

by $E \otimes \mathcal{U}$ and take the long exact sequence for sheaf cohomology.

$$h^0(E \otimes \mathcal{U}|_C) = 4k + 2 - d$$

where $d \in [0, 4]$ is the dimension of the image of

$$H^0(E \otimes \mathcal{U}|_{L_1}) \oplus H^0(E \otimes \mathcal{U}|_{L_2}) \rightarrow H^0(E \otimes \mathcal{U}|_P) \quad (111)$$

The following argument shows that $d = 4$ for the generic pair of intersecting L_1 and L_2 . First, each point P not in the closed orbit lies is the intersection of at least two distinct lines (lemma 2.24). For each such point P and each pair L_1, L_2 intersecting at P , we get two vectors e_1, e_2 in the fiber of E at P : the images of the canonical

$$\mathcal{O}_{L_i}(k) \rightarrow E|_{L_i}$$

Note that

$$\mathcal{U}|_{L_i}(k) \rightarrow E \otimes \mathcal{U}|_{L_i}$$

is surjective on global sections as $k > 0$. It follows that the map (111) is actually the restriction

$$e_1 \otimes H^0(\mathcal{U}|_{L_1}(k)) \oplus e_2 \otimes H^0(\mathcal{U}|_{L_2}(k)) \rightarrow E_P \otimes U_P$$

As $\mathcal{U}|_{L_i}(k)$ is generated by global sections for $k > 0$, (111) becomes

$$(e_1 \oplus e_2 \rightarrow E) \otimes U_P$$

so that its image is not 4-dimensional if and only if $e_1 = e_2$.

Finally, we are going to prove that if for the generic pair of lines the induced e_1, e_2 are equal, then there is a section of E defined on the complement of a curve.

The locus of lines whose order of jump is at least $k+1$ is at most a curve, so that its preimage inside the universal line \mathcal{L} is at most a surface S_E . On the complement of S_E we can define a section of r_Y^*E which (by flatness of r_Y) descends to a section of E on $r_Y(\mathcal{L} \setminus S_E)$, which is an open subvariety of Y as r_Y is flat.

Note that if $Y \setminus r_Y(\mathcal{L} \setminus S_E)$ contains a surface, then its preimage in \mathcal{L} is saturated with respect to both r_Y and r_A , contradicting the fact that for each two points we can find a chain of lines connecting them (as for any two lines there is a third one intersecting both of them by proposition 2.26).

To conclude, note that a section of E defined on a complement of a curve extends to a global section of E , contradicting the stability of E . \square

5 Minimal instantons

In this section we provide a complete description of the moduli space on instantons on Y in the case of $c_2 = 2$.

As explained in remark 4.2, for any instanton E we have

$$h^1(E) = c_2(E) - 2 = n - 2$$

As a consequence, there are no instantons with $c_2(E) < 2$.

Definition 5.1. Instantons such that $c_2(E)$ takes the minimal possible value, i.e. $c_2(E) = 2$ are called minimal instantons.

The first result that we prove about minimal instantons is the following consequence of theorem 4.17.

Proposition 5.2. *For a minimal instanton E the generic line is not jumping. Moreover, the support of JL_E is a smooth conic.*

Proof. We prove directly the second part of the statement, which includes the first. From [Kuz12, Prop. 4.10] we know that the support of JL_E is $\gamma^{-1}(\Delta_H)$, where Δ_H is the discriminant inside $\mathbb{P}(S^2H^*)$ and

$$\gamma : A \cong A^* \rightarrow S^2H^*$$

is any preimage of E via the projection from \mathbf{M}_n to \mathcal{MI}_2 .

By 4.17 γ is injective. In the case of minimal instantons

$$\dim(A) = \dim(S^2H^*) = 3$$

so that γ is also surjective. As γ is an isomorphism and Δ_H is a smooth conic in $\mathbb{P}(S^2H^*)$, the jumping divisor of a minimal instanton is a smooth conic. \square

The main result in this section is theorem 5.8. It provides an SL_2 -equivariant open embedding of \mathcal{MI}_2 inside $\mathbb{P}(S^2(A^*))$ and a complete SL_2 -equivariant description of the complement $\mathbb{P}(S^2 A^*) \setminus \mathcal{MI}_2$.

One of the key steps in the proof of theorem 5.8 is that in the case of minimal instantons, the conditions of theorem 4.8 specialize to a simpler single condition, as explained in lemma 5.3.

Lemma 5.3. *In the case $n = 2$, condition 4.8 (i) is equivalent to condition 4.8 (ii).*

Proof. Assume that γ is a fiberwise monomorphism. Then the quotient

$$0 \rightarrow H \otimes \mathcal{U} \xrightarrow{\gamma'} H^* \otimes \mathcal{U}^\perp \rightarrow F \rightarrow 0$$

is a vector bundle of rank 2 with $c_1(F) = 0$. Moreover $H^\bullet(F) = 0$ as the other two bundles are acyclic.

We dualize and we find

$$0 \rightarrow F \rightarrow H \otimes V/\mathcal{U} \xrightarrow{-\gamma'^*} H^* \otimes \mathcal{U}^* \rightarrow 0$$

Taking long exact sequence in cohomology and substituting $H^\bullet(F) = 0$ we find that γ' induces an isomorphism

$$H \otimes V \xrightarrow{-\hat{\gamma}} H^* \otimes V^*$$

which is condition (ii).

The other way round, assume $\text{rank}(\hat{\gamma}) = 10$, i.e. assume $\hat{\gamma}$ is an isomorphism, and restrict diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H \otimes \mathcal{U} & \longrightarrow & H \otimes V \otimes \mathcal{O}_Y \\ & & \downarrow \gamma' & & \downarrow \hat{\gamma} \\ 0 & \longrightarrow & H^* \otimes \mathcal{U}^\perp & \longrightarrow & H^* \otimes V^* \otimes \mathcal{O}_Y \end{array}$$

to any closed point U to find

$$\begin{array}{ccccc} 0 & \longrightarrow & H \otimes U & \longrightarrow & H \otimes V \\ & & \downarrow \gamma'|_U & & \downarrow \hat{\gamma} \\ 0 & \longrightarrow & H^* \otimes U^\perp & \longrightarrow & H^* \otimes V^* \end{array} \tag{112}$$

As the map $\hat{\gamma}$ is injective, the map $\gamma'|_U$ is injective as well, so that condition (i) holds. \square

The correspondence between instantons and curves with theta-characteristics takes a very simple form in the minimal case, as on smooth conics there's a unique choice of a theta-characteristic. We want to use this correspondence to give a complete description of \mathcal{MI}_2 inside $\mathbb{P}(S^2 A^*)$.

Let us first recall that, by theorem 4.14, \mathcal{MI}_2 embeds in $\mathbb{P}(S^2 A^*)$ by sending an instanton E to its curve of jumping lines, which is a conic in $\mathbb{P}(A)$ and can be therefore thought of as an element of $\mathbb{P}(S^2 A^*)$.

Proof. (Sketch) Given a minimal instanton E one can obtain a conic in $\mathbb{P}(A)$ via theorem 4.14. Conversely, given a conic C

$$i_C : C \rightarrow \mathbb{P}(A)$$

which comes from an instanton via theorem 4.14, take the pushforward $i_{C*}\mathcal{O}_C(-1)$ and decompose it with respect to the exceptional collection

$$\langle \mathcal{O}_{\mathbb{P}(A)}(-2), \mathcal{O}_{\mathbb{P}(A)}(-1), \mathcal{O}_{\mathbb{P}(A)} \rangle$$

This will provide a resolution

$$0 \rightarrow H \otimes \mathcal{O}_{\mathbb{P}(A)}(-2) \rightarrow H^* \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \rightarrow i_{C*}\mathcal{O}_C(-1) \rightarrow 0$$

Finally, one checks that such a resolution is symmetric in H and recovers in this way an element in $A^* \otimes S^2 H^*$ corresponding to an instanton. \square

In section 2.1 we have discussed an $SL(W)$ -equivariant construction of Y : its first step is the choice of an $SL(W)$ -invariant 3-dimensional $A \subset \Lambda^2 V^*$. As $SL(W)$ acts on A , it also acts on $A \otimes S^2 H^*$. In the case of minimal instantons there is another copy of SL_2 acting on $A \otimes S^2 H^*$, namely $SL(H)$.

As the two actions of $SL(W)$ and $SL(H)$ on $A \otimes S^2 H^*$ commute, they induce an action of $SL(W) \times SL(H)$. The description of the complement of the locus of instantonic γ in $A \otimes S^2 H^*$ involves the two $SL(W) \times SL(H)$ -invariant divisors which we are about to introduce.

Definition 5.4. We denote the degree 3 divisor in $\mathbb{P}(A \otimes S^2 H^*)$ of degenerate maps from A^* to $S^2 H^*$ by Q_3 .

There is a unique divisor of degree 2 in $\mathbb{P}(A \otimes S^2 H^*)$ which is invariant under the action of $SL(W) \times SL(H)$. We denote it by Q_2 .

Remark 5.5. Let us show that Q_2 is well defined. It is easy to determine the irreducible components of each direct summand of

$$S^2(A^* \otimes S^2 H) = (S^2 A^* \otimes S^2 S^2 H) \oplus (\Lambda^2 A^* \otimes \Lambda^2 S^2 H)$$

with respect to the action of $SL(W) \times SL(H)$. They are all the products of irreducible components of the factors. It follows that there is only one 1-dimensional $SL(W) \times SL(H)$ irreducible representation, which shows that Q_2 is well defined. Note moreover that, as there are clearly no $SL(W) \times SL(H)$ invariant elements in $A^* \otimes S^2 H$, the divisor Q_2 is irreducible.

Proposition 5.6. *The complement $\partial M_2 = A \otimes S^2 H^* \setminus M_2$ is the union of Q_2 and Q_3 .*

Proof. By lemma 5.3 we know that $\gamma \in M_2$ if and only if it satisfies condition (ii). As $\hat{\gamma}$ is a 10 by 10 antisymmetric matrix, it has rank 10 if and only if its Pfaffian does not vanish. As the entries of $\hat{\gamma}$ are linear in the coordinates of $A \otimes S^2 H^*$, ∂M_2 is cut by a

degree 5 equation. Moreover, by 4.17 we know that for γ to be in M_2 it is necessary that it induces an isomorphism from A^* to S^2H^* , so that Q_3 is contained in ∂M_2 .

As Q_3 has degree 3, we are left with the task of finding a missing quadric, which we claim is Q_2 . To prove it, we use the joint action of $SL(W)$ and $SL(H)$. The instantonic conditions (i), (ii) are clearly invariant under the action of both copies of SL_2 , so that also the missing quadric has the same property. By remark 5.5, having degree 2 and being $SL(W) \times SL(H)$ -invariant uniquely determine Q_2 , so that finally ∂M_2 is the union of Q_2 and Q_3 . \square

Theorem 5.8 completes the description of \mathcal{MT}_2 inside $\mathbb{P}(S^2A^*)$. In order to state it, we need to choose a notation for two divisors in $\mathbb{P}(S^2A^*)$. We do it in the following definition.

Definition 5.7. We denote the degree 3 divisor in $\mathbb{P}(S^2A^*)$ of degenerate conics in $\mathbb{P}(A)$ by Δ_A .

There is a unique hyperplane in $\mathbb{P}(S^2A^*)$ which is invariant under the action of $SL(W)$. We denote it by H_q as under the canonical $S^2A^* \cong S^2A$ it is cut by q .

Denote by F the rational surjection

$$F : \mathbb{P}(A \otimes S^2H^*) \dashrightarrow \mathbb{P}(S^2A^*)$$

sending γ to $\gamma^{-1}(\Delta_H)$. Note that F has degree 2 and that by the decomposition of $S^2(A \otimes S^2H^*)$ and S^2A^* into $SL(W) \times SL(H)$ -irreducibles it is the unique $SL(W) \times SL(H)$ -equivariant map of degree 2 from $\mathbb{P}(A \otimes S^2H^*)$ to $\mathbb{P}(S^2A^*)$. Another way to describe F is to say that

$$F(\gamma) = \gamma^T \cdot q_H \cdot \gamma \quad (113)$$

where γ^T is the transpose of γ and $q_H \in S^2H^*$ is the unique $SL(H)$ -invariant.

We are now ready to state and prove theorem 5.8.

Theorem 5.8. *The map which associates with an instanton E of charge 2 its conic of jumping lines is an SL_2 -equivariant isomorphism*

$$\mathcal{MT}_2 \rightarrow \mathbb{P}(S^2A^*) \setminus (\Delta_A \cup H_q)$$

where Δ_A and H_q are defined in 5.7.

Proof. As F is surjective, it is enough to show that

$$F^{-1}(H_q) = Q_2 \quad F^{-1}(\Delta_A) = Q_3$$

The first one holds as F is $SL(H)$ -equivariant of degree 2 and H_q is $SL(W)$ -equivariant, so that $F^{-1}(H_q)$ is invariant under the action of both $SL(W)$ and $SL(H)$. By remark 5.5, the only such quadric is Q_2 .

As for the second one, by equation (113) we have

$$\det(F(\gamma)) = \det(\gamma)^2 \det(q_H)$$

so that $F(\gamma)$ is degenerate if and only if γ is degenerate. \square

A straightforward consequence of theorem 5.8 is the existence of a unique minimal instanton E_0 with an SL_2 -equivariant structure.

Corollary 5.9. *There is a unique minimal instanton E_0 with an SL_2 -equivariant structure.*

Proof. We have already discussed the existence and uniqueness of an SL_2 -invariant point in $q \in \mathbb{P}(S^2 A^*)$. By theorem 5.8, to show that q comes from an instanton it is enough to check that

$$q \notin H_q \cup \Delta_A$$

By definition of irreducible component, $q \notin H_q$. To conclude the proof, recall that Δ_A is the locus of degenerate conics in A while q is non-degenerate, as its kernel is an SL_2 -submodule of A . \square

Remark 5.10. It is interesting to play the 2-rays game on the projectivization $\mathbb{P}_Y(E_0)$ (see for example [Cor00]).

Remark 5.11. It is possible to combine proposition 4.29 and proposition 2.44 into another proof of the fact that for each minimal instanton the generic line is not jumping.

The argument is the following: if there is a minimal instanton E such that the generic line is jumping for it, then the support of JC_E contains the divisor of singular conics. This contradicts the fact that the degree of the support of JC_E is 2, while the degree of the locus of singular conics is 3.

The last fact we prove about \mathcal{MI}_2 is that it is not a fine moduli space.

Proposition 5.12. *There is no universal family of minimal instantons.*

Proof. Assume on the contrary there is a universal family \mathcal{E} of minimal instantons. Then, by theorem 4.14 there is a family $\Phi_{\mathcal{L}}(\mathcal{E}(-1))$ of smooth conics with a theta-characteristic. This means that if we consider over $\mathbb{P}(A) \times \mathbb{P}(S^2 A^*)$ the incidence divisor I , there is an open subset I° with a line bundle restricting to $\mathcal{O}(-1)$ on each fiber of the projection to $\mathbb{P}(S^2 A^*)$.

As I is a smooth divisor of degree $(2, 1)$, by Lefschetz hyperplane section theorem $\mathrm{Pic}(I)$ is generated by the restrictions of $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ from the ambient space. As I° is open in I , the same holds for I° . It follows that the restriction of any line bundle on I° to a fiber of the projection to $\mathbb{P}(S^2 A^*)$ has even degree. This provides a contradiction with the existence of \mathcal{E} . \square

6 Instantons of charge 3

In this section we are going to describe the moduli space \mathcal{MI}_3 of instantons of charge 3. More precisely, in 6.1 we construct a natural dominant map β from \mathcal{MI}_3 to $\mathrm{Gr}(3, V)$ and we introduce a class of instantons which we call special. After this, in 6.2 we lift the map β to an embedding into a relative Grassmannian. The embedding provides \mathcal{MI}_3 with a natural compactification. Finally, in 6.3 and 6.4 we treat the cases of special and of non-special instantons separately, focussing on the properties of their jumping lines.

6.1 A map from \mathcal{MI}_3 to $\mathbb{G}r(3, V)$

It turns out that there is a surjective map from \mathcal{MI}_3 to $\mathbb{G}r(3, V)$, which we will denote by β . Its fibers are either projective spaces or Grassmannians and it is convenient to stratify $\mathbb{G}r(3, V)$ with respect to their type. More precisely, there will be two strata and β will be smooth on each of them. This will correspond to a distinction between two kinds of instantons: the special ones and the non-special ones.

Notation 6.1. We will denote $\mathbb{G}r(3, V)$ by B . We will also denote $\kappa(A)$, that is to say the image of κ in $\mathbb{G}r(3, V)$, by B^s and its complement $B \setminus B^s$ by B^n .

Notation 6.2. When $c_2(E) = 3$, the space H' in monad (93) is 1-dimensional. For this reason we will write

$$H \otimes \mathcal{U} \rightarrow H^* \otimes \mathcal{U}^\perp \rightarrow \mathcal{O}_Y \quad (114)$$

for any monad associated with an instanton E of charge 3. The first map in the complex

$$H \xrightarrow{\gamma} H^* \otimes A \rightarrow V^* \quad (115)$$

obtained by applying $\text{Hom}(\mathcal{U}, -)$ to (114) is denoted by γ .

Lemma 6.3. *For a charge 3 instanton the following properties are equivalent*

- $\text{ext}^1(\mathcal{U}, E) \neq 0$
- $\text{hom}(\mathcal{U}, E) \neq 1$

Proof. We can compute both $\text{Hom}(\mathcal{U}, E)$ and $\text{Ext}^1(\mathcal{U}, E)$ using the monad (99): they are the cohomology of the complex (115). Note that γ is injective as its kernel would contribute to $\text{Ext}^{-1}(\mathcal{U}, E)$.

It follows from (115) that

$$\text{hom}(\mathcal{U}, E) - \text{ext}^1(\mathcal{U}, E) = 1$$

showing the equivalence of the two conditions. □

The above lemma motivates the following definition.

Definition 6.4. An instanton E of charge 3 is *special* if $\text{Ext}^1(\mathcal{U}, E) \neq 0$. Equivalently, an instanton is *non-special* if $\text{Hom}(\mathcal{U}, E) = \mathbb{C}$.

We will see that special instantons are special in many ways. Essentially by definition they do not have a canonical map into \mathcal{U}^* , while most of instantons have one. They are the only instantons having 2-jumping lines (see proposition 6.24). They are the only instantons whose associated theta-characteristic is not locally free (see proposition 7.3).

We are now going to introduce the main character in the description of \mathcal{MI}_3 .

Definition 6.5. The map

$$\beta : \mathcal{MI}_3 \rightarrow B$$

sends E to $\text{Ext}^1(-, E)$ of the tautological map $V^* \otimes \mathcal{O}_Y \rightarrow \mathcal{U}^*$.

So far it is not clear that

$$\mathrm{Ext}^1(\mathcal{U}^*, E) \rightarrow \mathrm{Ext}^1(V^* \otimes \mathcal{O}_Y, E) \cong V \otimes H^1(E) \cong V$$

is a 3-dimensional subspace of V and that β is regular: we prove it in proposition 6.7 by means of lemma 6.6.

Lemma 6.6. *If E is an instanton with $c_2(E) \geq 3$, then $\mathrm{Hom}(\mathcal{U}^\perp, E) = 0$.*

Proof. Both E and \mathcal{U}^\perp are μ -stable (see lemma 2.16) of slope respectively $\mu(E) = 0$ and $\mu(\mathcal{U}^\perp) = -1/3$. It follows that there is no map of rank 1 from \mathcal{U}^\perp to E , as otherwise

$$\mu(E) > \mu(\mathrm{Im}) > \mu(\mathcal{U}^\perp)$$

which contradicts the fact that the slope of the image $\mu(\mathrm{Im})$ is an integer.

As E is torsion-free, there are no maps of rank 0, so that we only have to check that there are no maps of rank 2. To prove it, assume there is such a map and complete it to an exact sequence.

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{U}^\perp \rightarrow E \rightarrow Q \rightarrow 0$$

Then \mathcal{L} is reflexive of rank 1 and therefore it is a line bundle. Moreover, $c_1(\mathcal{L}) \leq -1$ by stability of \mathcal{U}^\perp and $c_1(\mathcal{L}) \geq -1$ by stability of E , so that $\mathcal{L} \cong \mathcal{O}(-1)$. To conclude the proof, we use equations (28) and (91) to compute $\mathrm{ch}(Q)$ and derive a contradiction:

$$\mathrm{ch}(Q) = \mathrm{ch}(E) - \mathrm{ch}(\mathcal{U}^\perp) + \mathrm{ch}(\mathcal{O}(-1))$$

yields

$$\mathrm{ch}(Q) = (3L - c_2(E)) - P$$

so that the leading term is negative when $c_2(E) \geq 3$. □

The bound in lemma 6.6 is sharp as instantons of charge 2, by monadic description, always have a 2-dimensional space of maps from \mathcal{U}^\perp .

Proposition 6.7. *β is a regular map from \mathcal{MI}_3 to B .*

Proof. By self duality of E , $\mathrm{ext}^1(\mathcal{U}^*, E) = \mathrm{ext}^1(E, \mathcal{U})$, and by monadic description (93) we have $\mathrm{ext}^1(E, \mathcal{U}) = 3$. Moreover the kernel of $\mathrm{Ext}^1(\mathcal{U}^*, E) \rightarrow V \otimes H^1(E)$ sits inside $\mathrm{Hom}(\mathcal{U}^\perp, E)$, which vanishes by lemma 6.6, so that

$$\mathrm{Ext}^1(\mathcal{U}^*, E) \subset V \otimes H^1(E)$$

gives a point in B .

We prove that the map β is regular by constructing it in families. Take any flat family \mathcal{E} of instantons of charge 3 over a base scheme S . Take the tautological map $V^* \otimes \mathcal{O}_Y \rightarrow \mathcal{U}^*$ and out of it construct

$$\mathrm{Hom}(\mathcal{U}^*, \mathcal{E}) \rightarrow V \otimes \mathrm{Hom}(\mathcal{O}_Y, \mathcal{E})$$

Push it forward to S . By cohomology and base change, we obtain a (shifted by 1) rank 3 subbundle of $V \otimes \mathcal{H}^1(E)$, where $\mathcal{H}^1(\mathcal{E})$ is a line bundle. By twisting the subbundle by $\mathcal{H}^1(\mathcal{E})^*$ we finally get a rank 3 subbundle of $V \otimes \mathcal{O}_S$, which by defining property of B gives a map from S to B . By the universal property for the coarse moduli space of instantons, we induce a unique map from \mathcal{MI}_3 to B . \square

Lemma 6.8 and proposition 6.9 deal with different descriptions of β .

Lemma 6.8. $\beta(E)$ is the 3-dimensional subspace of V given by $\text{Ext}^1(E, -)$ applied to the tautological $\mathcal{U} \rightarrow V \otimes \mathcal{O}_Y$.

Proof. By self-duality of E , there is a functorial isomorphism between $\text{Ext}^1(-, E)$ and $\text{Ext}^1(E, -^*)$ inducing a commutative square

$$\begin{array}{ccc} \text{Ext}^1(E, V \otimes \mathcal{O}) & \longleftarrow & \text{Ext}^1(E, \mathcal{U}) \\ \downarrow \cong & & \downarrow \cong \\ \text{Ext}^1(V^* \otimes \mathcal{O}, E) & \longleftarrow & \text{Ext}^1(\mathcal{U}^*, E) \end{array} \quad (116)$$

\square

Proposition 6.9. Let (114) be a monad for E . $\beta(E)$ is the image of the injective map

$$H^* \cong \text{Hom}(\mathcal{U}^\perp, H^* \otimes \mathcal{U}^\perp) \rightarrow \text{Hom}(\mathcal{U}^\perp, \mathcal{O}_Y) \cong V$$

obtained from (114) by applying $\text{Hom}(\mathcal{U}^\perp, -)$.

Proof. Under the canonical identification of B with the Grassmannian of 2-dimensional quotients of V , $\beta(E)$ is the bottom row the the following commutative diagram.

$$\begin{array}{ccccc} \text{Hom}(\mathcal{U}^\perp, H^* \otimes \mathcal{U}^\perp) & \xrightarrow{\cong} & \text{Hom}(\mathcal{U}^\perp, \tilde{E}) & \xleftarrow{0} & 0 = \text{Hom}(\mathcal{O}_Y, \tilde{E}) \\ & \searrow & \downarrow & & \downarrow 0 \\ & & \text{Hom}(\mathcal{U}^\perp, \mathcal{O}_Y) & \xleftarrow{\cong} & V \otimes H^0(\mathcal{O}_Y) \\ & & \downarrow & & \downarrow \cong \\ & & \text{Ext}^1(\mathcal{U}^\perp, E) & \xleftarrow{\beta(E)} & V \otimes H^1(E) \end{array} \quad (117)$$

The upper left triangle is induced by the monad (114), the middle and right column are induced by the acyclic extension sequence (92). The horizontal maps are induced by $\mathcal{U}^\perp \rightarrow V^* \otimes \mathcal{O}_Y$. The commutativity of (117) shows that

$$\text{Hom}(\mathcal{U}^\perp, H^* \otimes \mathcal{U}^\perp) \rightarrow \text{Hom}(\mathcal{U}^\perp, \mathcal{O}_Y)$$

is the kernel of $\beta(E)$. \square

Notation 6.10. By proposition 6.9, for a family of instantons \mathcal{E} over a base S we will write

$$\mathcal{H} \boxtimes \mathcal{U} \rightarrow \beta^* \mathcal{K} \boxtimes \mathcal{U}^\perp \rightarrow \mathcal{O}_{S \times Y} \quad (118)$$

for its decomposition with respect to the usual collection (35).

Next, we characterize the special instantons in terms of their image under β . Recall that we have introduced a stratification $\{B^s, B^n\}$ for $B = \mathbb{G}r(3, V)$, and that the close stratum B^s is the image under κ of $\mathbb{P}(A)$.

Proposition 6.11. *An instanton E is special if and only if $\beta(E) \in B^s$.*

Proof. E is special if and only if in the sequence (115) the map

$$H^* \otimes A \xrightarrow{\epsilon} V^*$$

is not surjective. Instead of computing $\text{Ext}^1(-, E)$ only on \mathcal{U} , compute it on the universal map $\mathcal{U} \rightarrow A^* \otimes \mathcal{U}^\perp$. The result is a commutative diagram

$$\begin{array}{ccc} A \otimes H^* & \xrightarrow{\quad \epsilon \quad} & V^* \\ \uparrow \text{Id} \cong & & \uparrow a(v, -) \\ A \otimes H^* & \xrightarrow{A \otimes \beta(E)} & A \otimes V \end{array}$$

showing that ϵ has a cokernel if and only if there is $v \in V$ such that $A(\beta(E), v) = 0$. As $\dim \beta(E) = 3$, this can happen if and only if $\dim A(v, -) \leq 2$, which is the case if and only if there is $a_v \in A$ such that $v = \ker a_v$. Finally, if $v = \ker a_v$, there's a unique choice for $\beta(E)$ and it is $\kappa(a_v)$. \square

The following is a more accurate description of the moduli space \mathcal{MT}_3^s of special instantons. Note that so far we have defined what a single special instanton is, but we do not have a notion of family of special instantons.

Let S be a scheme and denote the projections from $Y \times S$ to Y and S respectively by π_Y and π_S .

Definition 6.12. We will denote by \mathcal{MT}_3^s the functor which associates with S the set of families of instantons \mathcal{E} such that the schematic support $R^1 \pi_{\mathcal{MT}_3^*} \mathcal{H}om(\pi_Y^* \mathcal{U}, \mathcal{E})$ is the whole S .

Remark 6.13. The formation of $R^1 \pi_{\mathcal{MT}_3^*} \mathcal{H}om(\pi_Y^* \mathcal{U}, \mathcal{E})$ commutes with arbitrary base changes $T \rightarrow S$. The reason is that by monad (114) all higher pushforwards

$$R^{>1} \pi_{\mathcal{MT}_3^*} \mathcal{H}om(\pi_Y^* \mathcal{U}, \mathcal{E})$$

vanish.

Remark 6.14. Also the formation of the support of $R^1\pi_{\mathcal{MI}_3*}\mathcal{H}om(\pi_Y^*\mathcal{U}, \mathcal{E})$ commutes with arbitrary base change. This is true as the rank of $R^1\pi_{\mathcal{MI}_3*}\mathcal{H}om(\pi_Y^*\mathcal{U}, \mathcal{E})$ is never greater than 1, so that its schematic support is the first degeneracy locus of the vector bundle map

$$A \otimes \beta^*\mathcal{K} \rightarrow V^* \otimes \mathcal{O}_S \quad (119)$$

To conclude, note that the formation of the above map commutes with arbitrary base change by functoriality of the decomposition with respect to a full exceptional collection.

Proposition 6.15. *There is a cartesian diagram*

$$\begin{array}{ccc} \mathcal{MI}_3^s & \hookrightarrow & \mathcal{MI}_3 \\ \beta^s \downarrow & \square & \downarrow \beta \\ \mathbb{P}(A) & \xrightarrow{\kappa} & B \end{array} \quad (120)$$

where the embedding of \mathcal{MI}_3^s in \mathcal{MI}_3 is the natural one.

Proof. Given a family of special instantons \mathcal{E} on a scheme S , consider its associated monad (118). Recall that $\beta(\mathcal{E})$ is by definition $\beta^*\mathcal{K} \rightarrow V \otimes \mathcal{O}_S$, whose cokernel is, by definition of family of special instantons, supported everywhere on S . It follows that the map from S to B factors uniquely via the degeneracy locus of (119), which by proposition 3.6 is B^s . As a consequence, we have constructed the maps in (120).

In the other direction, given a family of instantons \mathcal{E} over S such that $\beta(\mathcal{E})$ factors via B^s , we clearly have that

$$R^1\pi_{\mathcal{MI}_3*}\mathcal{H}om(\pi_Y^*\mathcal{U}, \mathcal{E})$$

is supported on the whole S as it comes by pullback from B^s , where by 3.6 it is a line bundle. \square

By definition 6.4, any non-special instanton has a canonical map $E \rightarrow \mathcal{U}^*$. By stability of E and \mathcal{U}^* , this map is injective, so that we have an exact sequence

$$0 \rightarrow E \rightarrow \mathcal{U}^* \rightarrow Q_E \rightarrow 0$$

where Q_E is by definition the cokernel of $E \rightarrow \mathcal{U}^*$. The Chern character computation yields

$$ch(Q_E) = H + \frac{7L}{2} - \frac{P}{6} \quad (121)$$

so that Q_E is supported on a linear section of Y . Later on, in theorem 6.34, we will prove that for any non-special E such a Q_E is always of the form $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2))$. In view of this, we prove that if Q_E is isomorphic to $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2))$, then $\beta(E) = K$.

Proposition 6.16. *For every short exact sequence*

$$0 \rightarrow E \rightarrow \mathcal{U}^* \rightarrow \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2)) \rightarrow 0$$

where E is an instanton and $K \in B \setminus D_{tri}$, it is true that $\beta(E) = K$.

Proof. We will compute $\beta(E)$ using the slight modification of definition 6.5 which is described in lemma 6.8.

We need to compute $\text{Ext}^1(E, -)$ on the tautological sequence

$$0 \rightarrow \mathcal{U} \rightarrow V \otimes \mathcal{O}_Y \rightarrow V/\mathcal{U} \rightarrow 0$$

As \mathcal{U}^* lies in the left orthogonal to all bundles in the sequence, we get isomorphisms

$$\text{Ext}^1(E, -) = \text{Ext}^2(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2)), -) = \text{Ext}^2(\mathcal{O}_{\mathbb{P}(K)}(2), \Phi_{\text{Bl}}^!(-)) \quad (122)$$

Our next goal is to check that the image of

$$\text{Ext}^2(\mathcal{O}_{\mathbb{P}(K)}(2), \Phi_{\text{Bl}}^!(\mathcal{U})) \rightarrow V \otimes \text{Ext}^2(\mathcal{O}_{\mathbb{P}(K)}(2), \Phi_{\text{Bl}}^!(\mathcal{O}_Y)) \quad (123)$$

is K . We will apply $\Phi_{\text{Bl}}^!$ to the tautological injection $\mathcal{U} \rightarrow V \otimes \mathcal{O}_Y$. In order to do it, recall the identifications of lemma 3.26. Note also that over $\mathbb{P}_Y(\mathcal{U})$ there is a commutative diagram

$$\begin{array}{ccc} & \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-E) & \\ & \searrow & \downarrow \epsilon_V(-E) \\ p_Y^* \mathcal{U}(h-E) & \longrightarrow & V \otimes \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h-E) \end{array}$$

where the diagonal map is the twisted relative tautological injection on $\mathbb{P}_Y(\mathcal{U})$ and the horizontal map is the pullback of the tautological injection on Y . If we push the diagram forward to $\mathbb{P}(V)$ we find another commutative diagram

$$\begin{array}{ccc} & I_{\sigma(A)} & \\ & \searrow & \downarrow \epsilon_V(-E) \\ \Phi_{\text{Bl}}^!(\mathcal{U})[-1] & \longrightarrow & V \otimes \Phi_{\text{Bl}}^!(\mathcal{O}_Y)[-1] \end{array} \quad (124)$$

To find the map (123), apply the functor $\text{Ext}^2(\mathcal{O}_{\mathbb{P}(K)}(2), -)[1]$ to the above diagram. By Grothendieck duality

$$\text{Ext}^2(\mathcal{O}_{\mathbb{P}(K)}(2), -)[1] \cong H_{\mathbb{P}(K)}^1(Lj_K^*(-))$$

Applying $H_{\mathbb{P}(K)}^1(Lj_K^*(-))$ to diagram (124). The result is

$$\begin{array}{ccc} & H^1(Lj_K^* I_{\sigma(A)}) & \\ & \searrow \cong & \downarrow H^1(\epsilon_V) \\ H^1(Lj_K^* \Phi_{\text{Bl}}^!(\mathcal{U})[-1]) & \xrightarrow{\beta(E)} & V \otimes H^1(Lj_K^* I_{\sigma(A)}(h)) \end{array} \quad (125)$$

where the diagonal arrow is an isomorphism as its cokernel is $\mathcal{O}(-h)$ is acyclic on $\mathbb{P}(K)$. It follows that the image of $\beta(E)$ coincides with the image of $H^1(\epsilon_V)$.

As we assumed $K \notin D_{tri}$, we know by lemma 3.3 and lemma 3.21 that the intersection of $\mathbb{P}(K)$ with $\sigma(A)$ has expected dimension, so that $I_{\sigma(A)}$ pulls back to an ideal sheaf on $\mathbb{P}(K)$. More precisely, $Lj_K^* I_{\sigma(A)}$ is the ideal of a length 4 subscheme of $\mathbb{P}(K)$. We will denote $\mathbb{P}(K) \cap \sigma(A)$ by K_4 and its ideal in $\mathbb{P}(K)$ by I_4 , so that the vertical arrow in the diagram (125) becomes

$$H_{\mathbb{P}(K)}^1(\mathfrak{e}_V) : H^1(I_4) \rightarrow V \otimes H^1(I_4(h)) \quad (126)$$

Note that the restriction to $\mathbb{P}(K)$ of the Euler map \mathfrak{e}_V factors via the Euler map \mathfrak{e}_K of K :

$$\begin{array}{ccc} H^1(I_4) & & \\ \downarrow H^1(\mathfrak{e}_K) & \searrow H^1(\mathfrak{e}_V) & \\ K \otimes H^1(I_4(h)) & \xrightarrow{j_K} & V \otimes H^1(I_4(h)) \end{array}$$

In order to conclude the proof, we only need to check that $H^1(\mathfrak{e}_K)$ is an isomorphism. As the cokernel of the Euler map is $\mathcal{T}_{\mathbb{P}(K)}$, if

$$H^1(\mathcal{T}_{\mathbb{P}(K)} \otimes I_4) = H^2(\mathcal{T}_{\mathbb{P}(K)} \otimes I_4) = 0 \quad (127)$$

then $H^1(\mathfrak{e}_K)$ is an isomorphism.

To check this last vanishing, we use a resolution for I_4 . Using the fact that $\mathbb{P}(K) \notin D_{tri}$, we will prove that K_4 is always cut by two conics, so that I_4 has a Koszul resolution

$$0 \rightarrow \mathcal{O}(-4) \rightarrow \mathcal{O}(-2)^{\oplus 2} \rightarrow I_4 \rightarrow 0$$

For any scheme of length 4 in $\mathbb{P}(K)$ there is at least a pencil of conics containing it. Take a pencil of conics in $\mathbb{P}(K)$ containing K_4 : if they do not have common components, then the intersection is K_4 , otherwise the conics in the pencil share a component. If they share a component, K_4 is contained in a line with an embedded point, and therefore intersects the line in a scheme of length at least 3, against the fact that $\mathbb{P}(K) \notin D_{tri}$.

Finally,

$$H^\bullet(\mathcal{T}_{\mathbb{P}(K)}(-2)) = H^\bullet(\mathcal{T}_{\mathbb{P}(K)}(-4)) = 0$$

imply that (127) is true. □

6.2 An embedding of \mathcal{MI}_3 in a relative Grassmannian

The monadic data for the construction of an instanton of charge 3 consist of two maps: one from $H \otimes \mathcal{U}$ to $H^* \otimes \mathcal{U}^\perp$, the other from $H^* \otimes \mathcal{U}^\perp$ to \mathcal{O}_Y . When we constructed the map β in section 6.1 we used only the second one. By taking also the first one into account we expect to construct an embedding of \mathcal{MI}_3 .

Recall that we are using notation 6.1, so that B is $\mathbb{G}r(3, V)$ and $\{B^n, B^s\}$ is a stratification of B . We will construct a map Γ from \mathcal{MI}_3 to $\mathbb{G}r_B(3, A \otimes \mathcal{K})$ which will

fit in a commutative diagram

$$\begin{array}{ccc} \mathcal{MT}_3 & \xrightarrow{\Gamma} & \mathrm{Gr}_B(3, A \otimes \mathcal{K}) \\ & \searrow \beta & \downarrow \pi_B \\ & & B \end{array}$$

where π_B is the canonical projection.

Let \mathcal{E} be a family of instantons parametrized by S . Denote by π_S and π_Y the projections from $S \times Y$ to S and Y . Denote by $\beta_{\mathcal{E}}$ the composition of the map $S \rightarrow \mathcal{MT}_3$ corresponding to \mathcal{E} with β . The relative Beilinson spectral sequence provides us with a monad for \mathcal{E} :

$$\mathcal{H} \boxtimes \mathcal{U} \xrightarrow{\gamma_{\mathcal{E}}} \beta_{\mathcal{E}}^* \mathcal{K} \boxtimes \mathcal{U}^{\perp} \xrightarrow{\beta_{\mathcal{E}}} \mathcal{O}_{S \times Y} \quad (128)$$

where \mathcal{K} is the tautological bundle on B (see notation 6.2).

Remark 6.17. In order to construct a map Γ from \mathcal{MT}_3 to $\mathrm{Gr}_B(3, A \otimes \mathcal{K})$ which is relative to B , it is enough to associate naturally to any family of instantons \mathcal{E} a rank 3 subbundle of $\beta_{\mathcal{E}}^* \mathcal{K} \otimes A$.

Lemma 6.18. *Given a family of instantons \mathcal{E} parametrized by S ,*

$$\pi_{S*} \mathrm{Hom}(\pi_Y^* \mathcal{U}, \mathcal{H} \boxtimes \mathcal{U}) \xrightarrow{\pi_{S*} \mathrm{Hom}(p_Y^* \mathcal{U}, \gamma)} \pi_{S*} \mathrm{Hom}(\pi_Y^* \mathcal{U}, \beta_{\mathcal{E}}^* \mathcal{K} \boxtimes \mathcal{U}^{\perp}) \quad (129)$$

is a rank 3 subbundle of $\beta_{\mathcal{E}}^ \mathcal{K} \otimes A$.*

Proof. First, note that by Künneth formula the map (129) becomes

$$\mathcal{H} \otimes \mathcal{O}_S \rightarrow \beta_{\mathcal{E}}^* \mathcal{K} \otimes A$$

Next, let $s \in S$ be a closed reduced point and E_s the corresponding instanton. By base change and flatness of p_S , it is enough to check that for each closed reduced point of S the map

$$\mathrm{Hom}(\mathcal{U}, \mathcal{H} \otimes \mathcal{U}) \rightarrow \mathrm{Hom}(\mathcal{U}, \beta(E_s) \otimes \mathcal{U}^{\perp}) \quad (130)$$

is injective. This holds as a non-trivial kernel of (130) contributes to $\mathrm{Hom}^{-1}(\mathcal{U}, E_s)$, which vanishes. \square

Definition 6.19. Γ takes a family of instantons \mathcal{E} to the rank 3 subbundle (129).

Our next goal is to describe the closure of the image of Γ . Denote by Z_3 the zero locus on $\mathrm{Gr}_B(3, A \otimes \mathcal{K})$ of the composition

$$\mathcal{T}_3^{rel} \rightarrow A \otimes \mathcal{K} \rightarrow A \otimes V \otimes \mathcal{O}_{\mathrm{Gr}_B(3, A \otimes \mathcal{K})} \rightarrow V^* \otimes \mathcal{O}_{\mathrm{Gr}_B(3, A \otimes \mathcal{K})} \quad (131)$$

where the first map is the relative tautological injection, the second one is the pullback of the tautological injection from B tensored by A and the third one is the evaluation of forms in A on V .

First, we describe Z_3 . By proposition 3.6 there is an exact sequence

$$0 \rightarrow \mathrm{Ker} \rightarrow A \otimes \mathcal{K} \rightarrow V^* \otimes \mathcal{O}_B \rightarrow \kappa_* (\mathcal{O}_{\mathbb{P}(A)}(2)) \rightarrow 0 \quad (132)$$

which defines a reflexive sheaf Ker .

Definition 6.20. The restriction of $\mathcal{K}er$ to B^n , that is to say the kernel of

$$A \otimes \mathcal{K}|_{B^n} \rightarrow V^* \otimes \mathcal{O}_{B^n}$$

is a rank 4 vector bundle over B^n denoted by $\mathcal{K}er^n$. The kernel of

$$A \otimes \mathcal{K}|_{B^s} \rightarrow V^* \otimes \mathcal{O}_{B^s}$$

is a rank 5 vector bundle over B^s denoted by $\mathcal{K}er^s$.

Lemma 6.21. *The restriction of Z_3 to B^n is $\mathbb{P}_{B^n}(\mathcal{K}er^{n*})$. The restriction of Z_3 to B^s is $\mathbb{G}r_{B^s}(3, \mathcal{K}er^s)$.*

Proof. The formation of zero loci of maps of vector bundles commutes with arbitrary base change. \square

Lemma 6.22. *Z_3 is the zero locus of a regular section of $\mathcal{T}_3^{rel*} \otimes V^*$. Moreover, Z_3 is irreducible and Γ factors via Z_3 .*

Proof. Z_3 is schematically cut by a section of the rank 15 vector bundle

$$\mathcal{T}_3^{rel} \otimes V$$

by its definition in (131). We only need to check that the section is regular, i.e. that $\text{codim}(Z_3) = 15$. As $\dim(\mathbb{G}r_B(3, A \otimes \mathcal{K})) = 24$, it is enough to check that $\dim(Z_3) \leq 9$.

In the notation of definition 6.20 and by lemma 6.21

$$Z_3 = \mathbb{P}_{B^n}(\mathcal{K}er^{n*}) \sqcup \mathbb{G}r_{B^s}(3, \mathcal{K}er^s)$$

This gives a stratification of Z_3 into an open 9-dimensional subscheme and a closed 8-dimensional subscheme, so that $\dim(Z_3) \leq 9$.

The 9-dimensional open subscheme is irreducible and its complement is 8-dimensional, so that, if Z_3 were reducible, one of its components would be at most 8-dimensional. This is impossible as Z_3 is cut by 15 equations inside a Cohen-Macaulay scheme of dimension 24. It follows that Z_3 is irreducible.

Finally, by definition of Γ , the pullback of $\mathcal{H}om(\mathcal{U}, -)$ of

$$\mathcal{T}_3^{rel} \rightarrow A \otimes \mathcal{K} \rightarrow V^* \otimes \mathcal{O}_{\mathbb{G}r_B(3, A \otimes \mathcal{K})}$$

is the relative monad (128). As the relative monad is a complex, Γ factors via Z_3 . \square

We are now going to prove that the map Γ is an embedding of $\mathcal{M}I_3$ into $\mathbb{G}r_B(3, A \otimes \mathcal{K})$ and to show that the closure of its image is Z_3 .

Theorem 6.23. *Γ is an embedding and the closure of its image is Z_3 .*

Proof. Over $\mathrm{Gr}_B(3, A \otimes \mathcal{K}) \times Y$ there are canonical maps

$$\mathcal{T}_3^{rel} \boxtimes \mathcal{U} \rightarrow \mathcal{K} \boxtimes \mathcal{U}^\perp \rightarrow \mathcal{O}_{\mathrm{Gr}_B(3, A \otimes \mathcal{K})} \boxtimes \mathcal{O}_Y \quad (133)$$

The first map is the composition

$$\mathcal{T}_3^{rel} \boxtimes \mathcal{U} \rightarrow A \otimes \mathcal{K} \boxtimes \mathcal{U} \rightarrow \mathcal{K} \boxtimes \mathcal{U}^\perp$$

where the first map is the relative tautological injection and the second is the unique SL_2 -equivariant one. The second map is the unique SL_2 -invariant element of

$$\mathrm{Hom}_{\mathrm{Gr}_B(3, A \otimes \mathcal{K})}(\mathcal{K} \boxtimes \mathcal{U}^\perp, \mathcal{O}_{\mathrm{Gr}_B(3, A \otimes \mathcal{K})} \boxtimes \mathcal{O}_Y) = V^* \otimes V$$

By definition of Z_3 we have that the restriction of (133) to $Z_3 \times Y$ is a complex. By theorem 4.5, the moduli space \mathcal{MT}_3 is non-empty. Given an instanton E , by definition of Γ the restriction of (133) to $\Gamma(E) \times Y$ is a monad for E . It follows that on a non-trivial open subset $Z_3^o \subset Z_3$, the only non-trivial cohomology of the complex (133) is the middle one, and that it is a family of instantons.

Summarizing, for any family of instantons \mathcal{E} over a base S , there is an induced map to Z_3^o given by $\Gamma(\mathcal{E})$. Conversely, for any map from S to Z_3^o the pullback of (133) is a family of instantons. \square

6.3 Special instantons

By proposition 6.15, the moduli space \mathcal{MT}_3^s of special instantons embeds in $\mathrm{Gr}_{B^s}(3, \mathrm{Ker}^s)$. In this section we are going to provide an instantonic interpretation for the projection to B^s and we are going to show that such a projection is surjective.

Proposition 6.24. *Given an instanton E and a line L , the order of jump of E at L is 2 if and only if $\beta(E) = \kappa(a_L)$. In particular, E has a 2-jumping line if and only if it is special.*

Proof. Note that by corollary 4.18 there are no jumping lines of order greater than 2, so that L is a 2-jumping line for E if and only if $H^1(E|_L) \neq 0$.

Recall from corollary 2.23 that

$$L = \mathbb{P}(\kappa(a_L)/\ker(a_L))$$

so that

$$\mathcal{U}^\perp|_L \cong (A(\ker a_L, -) \otimes \mathcal{O}_L) \oplus \mathcal{O}_L(-1)$$

It follows that if we restrict a monad (93) for E to L we get

$$H \otimes (\mathcal{O}_L \oplus \mathcal{O}_L(-1)) \rightarrow \beta(E) \otimes (A(\ker a_L, -) \otimes \mathcal{O}_L \oplus \mathcal{O}_L(-1)) \rightarrow \mathcal{O}_L$$

The only contribution to $H^1(E|_L)$ comes from the rightmost term, so that $H^1(E|_L)$ is not zero if and only if the composition

$$\beta(E) \otimes A(\ker a_L, -) \rightarrow V \otimes V^* \rightarrow \mathbb{C}$$

vanishes.

As $\beta(E)$ is 3-dimensional and $A(\ker a_L, -)$ is 2-dimensional, this happens if and only if $\kappa(a_L) = \beta(E)$. \square

Next, we will prove that for each line L there is an instanton E such that $\beta(E) = \kappa(L)$. By proposition 6.24 it is enough to show that L is a 2-jumping line for E .

Let L be a line in Y and let $K \subset V$ be a 3-dimensional space such that $\kappa(L) \in \mathbb{P}(K)$. By lemma 3.20, we can choose K such that Y_K , the linear section of Y induced by K , is smooth. Note that in this case, by lemma 3.28, Y_K is the blow up of $\mathbb{P}(K)$ in 4 distinct points such that no 3 of them lie on a line. Denote the 4 components of the exceptional divisor by L, L_1, L_2, L_3 . In this notation, the following proposition holds.

Proposition 6.25. *Given a surjective map*

$$\mathcal{U} \rightarrow \mathcal{O}_{Y_K}(h - 2L) \tag{134}$$

denote its kernel by $E(-1)$. Then E is an instanton and L is a 2-jumping line for E .

Proof. We need to prove that $H^\bullet(E(-1)) = 0$, that $\text{ch}(E) = 2 - 3L$, that E is locally free and that it is μ -stable.

The vanishing of $H^\bullet(E(-1))$ follows from 2.17 and from the exact sequence

$$0 \rightarrow \mathcal{O}_{Y_K}(-L) \rightarrow \mathcal{O}_{Y_K}(h - 2L) \rightarrow \mathcal{O}_{h-L}(-1) \rightarrow 0$$

which shows that $\mathcal{O}_{Y_K}(h - 2L)$ is acyclic.

The Chern character computation follows from (28) and from the fact that it is easy to compute the Euler characteristic of several twists of $\mathcal{O}_{Y_K}(h - 2L)$ by means of the identification of Y_K with the blow up of $\mathbb{P}(K)$ in 4 points such that no 3 of them are collinear.

The fact that E is locally free follows from the isomorphism

$$\text{Tor}_i(E, -) \cong \text{Tor}_{i+1}(\mathcal{O}_{Y_K}(h - 2L), -)$$

and from the fact that locally $\mathcal{O}_{Y_K}(h - 2L)$ is the structure sheaf of a Cartier divisor, so that $\text{Tor}_{>1}(\mathcal{O}_{Y_K}(h - 2L), -) = 0$.

The stability of E can be checked via Hoppe's criterion 1.22. In order to do it, it is enough to check that $H^0(E)$ vanishes. This is equivalent to the fact that

$$H^0(\mathcal{U}^*) \rightarrow H^0(\mathcal{O}_{Y_K}(h - 2L)) \tag{135}$$

is surjective. Note that the restriction map $\mathcal{U}^* \rightarrow \mathcal{U}_{Y_K}$ is an isomorphism in cohomology, as its kernel \mathcal{U} is acyclic, so that the surjectivity of (135) is equivalent to that of

$$H^0(\mathcal{U}^*|_{Y_K}) \rightarrow H^0(\mathcal{O}_{Y_K}(h - 2L)) \tag{136}$$

Finally, as $\mathcal{U}_{Y_K}^* \rightarrow \mathcal{O}_{Y_K}(h - 2L)$ is surjective, its kernel is a line bundle. More precisely, it is determined by its c_1 , which is

$$H - h - 2L = 2h + L - L_1 - L_2 - L_3$$

It follows that (136) is surjective if and only if $H^1(\mathcal{O}_{Y_K}(2h + L - L_1 - L_2 - L_3))$ vanishes. To conclude note that

$$H^1(\mathcal{O}_{Y_K}(2h + L - L_1 - L_2 - L_3)) = H^1(2h - L_1 - L_2 - L_3) = 0$$

Our last claim is that L is a 2-jumping line for E . Let us prove it. Note that the Euler characteristic of the derived tensor product of $\mathcal{O}_{Y_K}(h - 2L)$ with a line does not depend on the choice of the line, so that it is always 1. In the case of our line L we have

$$\mathcal{O}_L \otimes \mathcal{O}_{Y_K}(h - 2L) = \mathcal{O}_L(2)$$

which implies

$$\mathrm{Tor}_1(\mathcal{O}_L, \mathcal{O}_{Y_K}(h - 2L)) = \mathcal{O}_L(1)$$

It follows that the $\mathrm{Tor}(\mathcal{O}_L, -)$ exact sequence for the defining sequence (134) of $E(-1)$ begins with an injection

$$\mathcal{O}_L(1) \rightarrow E(-1)|_L$$

which shows that L is 2-jumping for E . \square

Lemma 6.26. *For any L there is a smooth linear section Y_K of Y such that $\sigma(a_L) \in \mathbb{P}(K)$ and that the generic map from \mathcal{U} to $\mathcal{O}_{Y_K}(h - 2L)$ is surjective.*

Proof. First, note that by lemma 3.20 the generic plane $\mathbb{P}(K)$ in $\mathbb{P}(V)$ through $\sigma(a_L)$ is transverse to $\sigma(A)$ and contains no trisecants, so that the induced section Y_K is smooth.

The space of maps from \mathcal{U} to $\mathcal{O}_{Y_K}(h - 2L)$ is a \mathbb{C}^4 . This can be shown by noting that by adjunction

$$\mathrm{Hom}(\mathcal{U}, \mathcal{O}_{Y_K}(h - 2L)) \cong \mathrm{Hom}(p_Y^* \mathcal{U}, \mathcal{O}_{S_K}(h - 2L))$$

and that the relative tautological sequence (73) induces an exact sequence

$$0 \rightarrow H^0(\mathcal{O}_{S_K}(3h - E - 2L)) \rightarrow \mathrm{Hom}(\mathcal{U}, \mathcal{O}_{Y_K}(h - 2L)) \rightarrow H^0(\mathcal{O}_{S_K}(2h - 2L)) \rightarrow \dots$$

By lemma 3.22, $Rp_{V*} \mathcal{O}_{S_K} \cong \mathcal{O}_{\mathbb{P}(K)}$, so that the cohomology groups in the above sequence are easily computed via projection formula.

Note that as Y_K is smooth $K \notin D_{tri}$, so that by lemma 3.28 a map in $\mathrm{Hom}(\mathcal{U}, \mathcal{O}_{Y_K}(h - 2L))$ is surjective if and only if the corresponding one in $\mathrm{Hom}(p_Y^* \mathcal{U}, \mathcal{O}_{S_K}(h - 2L))$ is surjective. Recall that S_K was defined in definition 3.14 and is $p_V^{-1} \mathbb{P}(K)$. We will show that the space of non-surjective maps is 2-dimensional inside the \mathbb{P}^3 of maps from $p_Y^* \mathcal{U}$ to $\mathcal{O}_{S_K}(h - 2L)$. Denote by ϕ a map from $p_Y^* \mathcal{U}$ to $\mathcal{O}_{S_K}(h - 2L)$. Consider the composition

$$\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-h) \rightarrow p_Y^* \mathcal{U} \xrightarrow{\phi} \mathcal{O}_{S_K}(h - 2L) \tag{137}$$

If it is zero, then ϕ factors via $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-2h + E)$. As $h^0(\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(3h - E - 2L)) = 1$, we can assume that (137) is non-zero, so that, as S_K is irreducible, it is also injective. Consider the diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-h) & \longrightarrow & p_Y^* \mathcal{U} & \longrightarrow & \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-2h + E) \\ \downarrow & & \downarrow \phi & & \downarrow \\ \mathcal{O}_{S_K}(h - 2L) & \xrightarrow{1} & \mathcal{O}_{S_K}(h - 2L) & \longrightarrow & 0 \end{array}$$

By snake lemma, the cokernel of ϕ is isomorphic to the cokernel of

$$\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-2h + E) \rightarrow \mathcal{O}_Z(h - 2L)$$

where Z is the zero locus of a section of $\mathcal{O}_{S_K}(2h - 2L)$. It follows that Z contains a $\mathbb{P}^1 \subset S_K$ linearly equivalent to $h - L$. As a consequence, ϕ factors via $\mathcal{O}_{S_K}(h - L)$. One can finally check that there is a \mathbb{P}^1 of maps from \mathcal{U} to $\mathcal{O}_{S_K}(-L)$ and that there is a \mathbb{P}^1 of maps from $\mathcal{O}_{S_K}(-L)$ to $\mathcal{O}_{S_K}(h - 2L)$, so that the space of non-surjective maps is the image of $\mathbb{P}^1 \times \mathbb{P}^1$, showing eventually that the generic ϕ is surjective. \square

6.4 Non-special instantons

The aim of this section is to construct a family of instantons corresponding to an open dense subset of \mathcal{MI}_3 . We will also prove that this family is the universal family for non-special instantons. Consistently with notation 6.1 and definition 6.12, we will denote the moduli space of non-special instantons by \mathcal{MI}_3^n .

Note that, by lemma 6.21 and theorem 6.23, we already have a description of the moduli space of non-special instantons as an open subset of $\mathbb{P}_{B^n}(\text{Ker}^{n*})$. We want to improve this description in two ways: by finding the image $\beta(\mathcal{MI}_3^n)$ inside B^n and by describing the complement of \mathcal{MI}_3^n inside $\mathbb{P}_{B^n}(\text{Ker}^{n*})$.

In the next proposition we will construct non-special instantons as kernels of surjective maps from \mathcal{U}^* to $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$.

Proposition 6.27. *If $K \notin D_{tri}$, then the kernel E of a surjective map from \mathcal{U}^* to $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ is a non-special instanton of charge 3.*

Proof. Assume the kernel E of a surjective map from \mathcal{U}^* to $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ is given. To prove that E is an instanton of charge 3, we need to prove that it is locally free, that it is stable, that $H^1(E(-1)) = 0$ and that $ch(E) = 2 - 3L$. We will denote the map $\mathcal{U}^* \rightarrow \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ by ϕ . After that, we will check that it is non-special.

E is locally free if and only if the Tor-dimension of E is 0, i.e. if for any $i > 0$ and any y closed point $\text{Tor}_i(\mathcal{O}_y, E) = 0$. The defining sequence

$$0 \rightarrow E \rightarrow \mathcal{U}^* \xrightarrow{\phi} \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)) \rightarrow 0$$

gives an isomorphism between $\text{Tor}_i(\mathcal{O}_y, E)$ and $\text{Tor}_{i+1}(\mathcal{O}_y, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))$. We compute the latter via resolution (88), so that for $i > 1$ we always get 0, while for $i = 1$ there might be a non-trivial result. More precisely, by lemma 3.25 the condition $\text{Tor}_2(\mathcal{O}_y, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))) \neq 0$ is equivalent to $K \in D_{tri}$, so that under the assumption $K \notin D_{tri}$ we have that E is locally free.

By Hoppe's criterion 1.22, E is stable if and only if $H^0(E) = 0$. To check $H^0(E) = 0$, we check that the map

$$H^0(\mathcal{U}^*) \xrightarrow{H^0(\phi)} H^0(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))) \quad (138)$$

is injective. By the Grothendieck spectral sequence for the composition of two pushforwards, we get that the above map is the same as

$$H^0(p_Y^* \mathcal{U}^*) \xrightarrow{H^0(\phi^{ad})} H^0(p_V^* \mathcal{O}_{\mathbb{P}(K)}(2)) \quad (139)$$

induced by the adjoint map ϕ^{ad} to ϕ , so that it is enough to check that $H^0(\phi^{ad})$ is injective.

As ϕ is surjective, so is ϕ^{ad} by lemma 6.30. Now we claim that the restriction map

$$p_Y^* \mathcal{U}^* \rightarrow p_Y^* \mathcal{U}^*|_{S_K} \quad (140)$$

is an isomorphism in cohomology. By Koszul resolution (79) this follows if we show that $p_Y^* \mathcal{U}^*(-2h)$ and $p_Y^* \mathcal{U}^*(-h)$ are acyclic. By twisting the tautological sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(2h - E) \rightarrow p_Y^* \mathcal{U}^* \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h) \rightarrow 0$$

by $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-2h)$ we get immediately that $p_Y^* \mathcal{U}^*(-2h)$ is acyclic. By twisting it by $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-h)$ we get a non trivial extension of $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}$ by $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)$, so that $p_Y^* \mathcal{U}^*(-h)$ is acyclic if and only if $h^1(\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)) = 1$ and its other cohomology groups vanish. The short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E) \rightarrow \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h) \rightarrow \mathcal{O}_E(h) \rightarrow 0$$

is SL_2 -equivariant and induces an SL_2 -equivariant cohomology long exact sequence

$$0 \rightarrow H^0(\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)) \rightarrow S^4 W \rightarrow S^4 W \oplus \mathbb{C} \rightarrow H^1(\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)) \rightarrow 0 \quad (141)$$

so that $h^1(\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)) = 1$. Summing up, we have proved that (140) is an isomorphism in cohomology.

In order to prove stability of E , we still need to prove that if

$$p_Y^* \mathcal{U}^*|_{S_K} \xrightarrow{\phi^{ad}|_{S_K}} p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h)$$

is surjective, then $H^0(\phi^{ad}|_{S_K})$ is injective, that is to say that its kernel has no global sections. As we have just proved that $\phi^{ad}|_{S_K}$ surjective, its kernel is locally free of rank one, and is determined by its c_1 . As $c_1(\mathcal{U}^*) = H = 3h - E$, our claim is now that $\mathcal{O}_{S_K}(h - E)$ has no global sections. This last fact can be reduced via resolution (79) to some vanishing in the cohomology of

$$\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-h - E), \quad \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(-E), \quad \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E)$$

It is easy to check that the first two line bundles are acyclic, and that $h^0(\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h - E))$, so that finally also $\mathcal{O}_{S_K}(h - E)$ has no global sections.

The vanishing of $H^1(E(-1))$ follows from

$$H^1(\mathcal{U}) = 0, \quad H^0(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))(-H)) = 0$$

While the vanishing of $H^1(\mathcal{U})$ is trivial, we check that

$$H^0(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))(-H)) = 0$$

Recall that $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ has a resolution (87). Twist it by $\mathcal{O}_Y(-H)$ and note that the only nontrivial vanishing is that of $H^\bullet(S^2\mathcal{U}^*(-H))$. On the other hand, the vanishing of the cohomology of $S^2\mathcal{U}^*(-H)$ is equivalent to the fact that \mathcal{U} is exceptional, which is in proposition 2.15. It follows that $E(-1)$ is acyclic.

The Chern character of E is $\text{ch}(\mathcal{U}^*) - \text{ch}(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))$. To check that it is $2-3L$ it is enough to substitute the Chern characters which are computed in table (28) and lemma 3.24.

Finally, the fact that E is non-special directly follows from proposition 6.16 and criterion 6.11, which tell us that $\beta(E) = K$ and that E is special if and only if $\beta(E) \in B^s$. By assumption $\beta(E) \notin D_{\text{tri}}$, while by lemma 3.21 $B^s \subset D_{\text{tri}}$, so that we have that E is not special. \square

Lemma 6.28. *For $i = 1, 2$, let E_i be the instanton associated (via proposition 6.27) to $\phi_i : \mathcal{U}^* \rightarrow \Phi_{\text{Bl}}(K_i)$, $K_i \notin D_{\text{tri}}$. If $E_1 \cong E_2$, then $K_1 = K_2$ and $\phi_1 = \phi_2$ up to rescaling.*

Proof. As $E_1 \cong E_2$, we have $K_1 = \beta(E_1) = \beta(E_2) = K_2$. As for ϕ_1 and ϕ_2 , there is a canonical arrow from $E_1 \cong E_2$ to \mathcal{U}^* , so that (up to rescaling f), we can find a diagram

$$\begin{array}{ccccc} E_1 & \longrightarrow & \mathcal{U}^* & \xrightarrow{\phi_1} & \Phi_{\text{Bl}}(K_1) \\ f \downarrow & & \downarrow 1 & & \downarrow \\ E_2 & \longrightarrow & \mathcal{U}^* & \xrightarrow{\phi_2} & \Phi_{\text{Bl}}(K_2) \end{array}$$

showing that ϕ_1 and ϕ_2 have the same kernel.

Moreover, by lemma 3.28 and as $K_i \notin D_{\text{tri}}$, the sheaf $\Phi_{\text{Bl}}(K_i)$ is a line bundle on its support. As its support is irreducible, the identity is its only automorphism up to rescaling. \square

We should also check that our construction produces at least one instanton. We will check it in lemma 6.31.

The next lemma, together with 6.31, will give us the dimension of the family of instantons which we have just constructed.

Lemma 6.29. *For any K , there is an exact sequence*

$$0 \rightarrow K^* \rightarrow \text{Hom}(\mathcal{U}^*, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))) \rightarrow \mathbb{C} \rightarrow 0 \quad (142)$$

Proof. Let Φ^* be the left adjoint functor to Φ_{Bl} . Then by definition there is a natural isomorphism

$$\text{Hom}_Y(\mathcal{U}^*, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))) \cong \text{Hom}_{\mathbb{P}(V)}(\Phi^*(\mathcal{U}^*), \mathcal{O}_{\mathbb{P}(K)}(2))$$

As $\Phi_{\text{Bl}} = Rp_{Y*} \circ Lp_V^*$, its left adjoint Φ^* is $Rp_{V*}(-\otimes \omega_{p_V}) \circ Lp_Y^*$. Substituting $\omega_{p_V} = \mathcal{O}(E)$ and using the tautological sequence (73) we get a distinguished triangle

$$Rp_{V*}\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(2h) \rightarrow \Phi^*(\mathcal{U}^*) \rightarrow Rp_{V*}\mathcal{O}(h+E) \quad (143)$$

The leftmost term is $\mathcal{O}_{\mathbb{P}(V)}(2h)$ by projection formula. The rightmost is computed by pushing forward the exact sequence

$$0 \rightarrow \mathcal{O}(h) \rightarrow \mathcal{O}(h+E) \rightarrow \mathcal{O}_E(h+E) \rightarrow 0 \quad (144)$$

to $\mathbb{P}(V)$. Note that for every point v in $\sigma(A)$, the sheaf $\mathcal{O}_E(h+E)$ is acyclic on the fiber $p_V^{-1}(v)$, as it is isomorphic to $\mathcal{O}_{p_V^{-1}(v)}(-1)$, so that

$$Rp_{V*}\mathcal{O}(h+E) \cong Rp_{V*}\mathcal{O}(h)$$

In the end, by taking the long exact sequence for (143) we have shown that there is a canonical exact sequence on $\mathbb{P}(V)$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(V)}(2h) \rightarrow \Phi^*(\mathcal{U}^*) \rightarrow \mathcal{O}_{\mathbb{P}(V)}(h) \rightarrow 0 \quad (145)$$

which after the application of $\text{Hom}(-, \mathcal{O}_{\mathbb{P}(K)}(2h))$ becomes sequence (142). \square

Now we want to identify surjective maps inside $\text{Hom}(\mathcal{U}^*, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))$. The next lemma allows us to do it without discussing the possible singularities of S_K and Y_K . Note that for any K there is an isomorphism

$$\text{Hom}_Y(\mathcal{U}^*, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))) \xrightarrow{\cong} \text{Hom}_{S_K}(p_{Y_K}^*\mathcal{U}^*, \mathcal{O}_{S_K}(2h)) \quad (146)$$

given by adjunction with respect to pullback and pushforward via the composition

$$S_K \xrightarrow{p_{Y_K}} Y_K \rightarrow Y \quad (147)$$

Lemma 6.30. *Let $K \notin D_{tri}$. A map in $\text{Hom}_Y(\mathcal{U}^*, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))$ is surjective the corresponding one in $\text{Hom}_{S_K}(p_{Y_K}^*\mathcal{U}^*, \mathcal{O}_{S_K}(2h))$ via (146) is surjective.*

Proof. By lemma 3.28, if $K \notin D_{tri}$ then the composition (147) is a closed embedding. Adjunction with respect to pullback and pushforward via closed embeddings clearly preserves surjectivity. \square

Lemma 6.31. *If $K \notin D_{tri}$, then the general map in $\text{Hom}_Y(\mathcal{U}^*, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))$ is surjective. The locus of non-surjective maps is a union of at most 5 hyperplanes.*

Proof. We will classify non-surjective maps and show that they form a divisor inside $\mathbb{P}(\text{Hom}_Y(\mathcal{U}, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))))$. By lemma 6.30, a map in $\mathbb{P}(\text{Hom}_Y(\mathcal{U}, \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))))$ is surjective if and only if the corresponding one in $\text{Hom}_{\mathbb{P}_Y(\mathcal{U})}(p_Y^*\mathcal{U}^*, p_V^*\mathcal{O}_{\mathbb{P}(K)}(2h))$ is surjective.

Take any map $f \in \text{Hom}_{\mathbb{P}_Y(\mathcal{U})}(p_Y^* \mathcal{U}^*, p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h))$: the composition with the tautological map $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(2h - E) \rightarrow \mathcal{U}^*$ is either 0 or not. If it vanishes, then f factors via $\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h)$. In this case f vanishes on a linear section of S_K , so that it is never surjective.

To check how many such maps there are, we compute $\text{hom}(\mathcal{O}_{S_K}(h), \mathcal{O}_{S_K}(2h)) = h^0(\mathcal{O}_{S_K}(h))$. By lemma 3.22, the blow-up projection $p_K : S_K \rightarrow \mathbb{P}(K)$ is such that the coevaluation map (84) is an isomorphism. Therefore, by projection formula for p_K , $H^\bullet(\mathcal{O}_{S_K}(h)) \cong H^\bullet(\mathcal{O}_{\mathbb{P}(K)}(h)) = K^*$, which means that maps factoring via $\mathcal{O}(h)$ form a hyperplane.

By lemma 3.3, whenever $K \notin B^s$ the fiber product S_K is irreducible. It follows that if the composition

$$\mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(2h - E) \rightarrow \mathcal{U}^* \xrightarrow{f} p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h)$$

does not vanish, then it is injective. As a consequence, on S_K there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(2h - E) & \longrightarrow & p_Y^* \mathcal{U}^* & \longrightarrow & \mathcal{O}_{\mathbb{P}_Y(\mathcal{U})}(h) \\ \downarrow & & \downarrow f & & \downarrow \\ p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h) & \xrightarrow{1} & p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h) & \longrightarrow & 0 \end{array}$$

By the snake lemma, there is an exact sequence

$$\mathcal{O}(h) \rightarrow \mathcal{O}_E(2h) \rightarrow \text{Coker } f \rightarrow 0$$

As h restricts trivially to E , we have

$$\mathcal{O}_E \rightarrow \mathcal{O}_E \rightarrow \text{Coker } f \rightarrow 0$$

E might be reducible or non-reduced, but in any case, if $\text{Coker } f \neq 0$ then there is a \mathbb{P}^1 denoted by e such that $\text{Coker } f \rightarrow \mathcal{O}_e$.

We want to show that for each such e there is a \mathbb{P}^2 of maps from $p_Y^* \mathcal{U}^*$ to $p_V^* \mathcal{O}_{\mathbb{P}(K)}(2h)$ which are not surjective on e_i , and thus factor via the twisted ideal $I_e(2h)$ of e in S_K . Note that e_i might not be a Cartier divisor inside S_K , so that we will write I_e instead of $\mathcal{O}_{S_K}(-e)$ for the ideal of e . To check $\text{hom}(p_Y^* \mathcal{U}^*, I_e(2h)) = 3$, use the long exact sequence induced by

$$0 \rightarrow I_e(2h) \rightarrow \mathcal{O}_{S_K}(2h) \rightarrow \mathcal{O}_e \rightarrow 0$$

We can easily check that $\text{hom}(p_Y^* \mathcal{U}^*, \mathcal{O}_e) = 1$ via the tautological sequence on S_K and using the fact that E restricts to e as -1 . Then we need to check that $\text{ext}^1(p_Y^* \mathcal{U}^*, I_e(2h)) = 0$. Again by tautological sequence it is enough to show that $H^1(\mathcal{O}_e(2h - E)) = H^1(\mathcal{O}_e(h)) = 0$, which is true as h restricts trivially to e , while E restricts as $\mathcal{O}_e(-1)$.

Finally, we have proven that the set of non-surjective maps is a union of at most 5 hyperplanes, such that each one represents maps that are non surjective on a divisor. The divisor where surjectivity fails can be either linearly equivalent to h or to one of the (at most 4) irreducible components e_i of E on S_K . \square

On the other hand, there are no surjective maps from \mathcal{U}^* to $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ for $K \in D_{\text{tri}}$, as proved in the following lemma.

Lemma 6.32. *For $K \in D_{tri}$ there are no surjective maps from \mathcal{U}^* to $\Phi_{Bl}(\mathcal{O}_{\mathbb{P}(K)}(2h))$*

Proof. We will show that if $\mathbb{P}(K)$ contains a trisecant L to $\sigma(A)$, then the fiber of $\Phi_{Bl}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ at the point $y \in Y$ corresponding to the trisecant via 3.12 is 3-dimensional.

By base change and as p_Y has relative dimension 1, the dimension of the restriction of $\Phi_{Bl}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ to y is greater than the dimension of

$$H^0\left(p_Y^{-1}(y), p_Y^* \mathcal{O}_{\mathbb{P}(K)}(2h)|_{p_Y^{-1}(y)}\right)$$

which is isomorphic to $H^0(\mathcal{O}_L(2))$. As the rank of \mathcal{U}^* is 2, it follows that there are no surjective maps from \mathcal{U}^* to $\Phi_{Bl}(\mathcal{O}_{\mathbb{P}(K)}(2h))$. \square

By performing the construction in 6.27 in families, we are now going to construct a 9-dimensional family of instantons and to prove that it is the universal family for the moduli space of non special instantons.

First, recall from definition 3.14 that over B , one can construct a universal family \mathcal{S} for S_K .

By lemma 6.29 the pushforward of $\mathcal{H}om_{B \times Y}(\mathcal{U}^*, R p_{Y*} \mathcal{O}_{\mathcal{S}}(2h))$ to B is a rank 4 vector bundle. Denote it by \mathcal{G} . It turns out that we have already discussed this vector bundle (or at least its restriction to the complement of D_{tri}): it is the bundle

Lemma 6.33. *There is an isomorphism of vector bundles*

$$\mathcal{K}er^{B \setminus D_{tri}} \cong \mathcal{G}^*|_{B \setminus D_{tri}}$$

up to a twist by a line bundle on $B \setminus D_{tri}$.

Proof. First, note that by Grothendieck duality for the embedding of Y_K in Y there is an isomorphism

$$\mathcal{H}om(\mathcal{U}^*, \mathcal{O}_{Y_K}(2h)) \cong \mathcal{E}xt^1(\mathcal{O}_{Y_K}(h - E), \mathcal{U})$$

which can be constructed in family, up to a twist by a line bundle on the base of the family. Recall also that by lemma 3.28, the map p_{Y_K} is an isomorphism between S_K and Y_K whenever $K \notin D_{tri}$.

Next, compute the decomposition of $\mathcal{O}_{Y_K}(h - E)$ with respect to the collection $\mathcal{U}, \mathcal{U}^\perp, \mathcal{O}_Y, \mathcal{O}_Y(H)$. As $H = 3h - E$, $\mathcal{O}_{Y_K}(h - E)$ lies in the right orthogonal to $\mathcal{O}_Y(H)$. Also the component with respect to \mathcal{O}_Y is easily computed:

$$\mathcal{R}Hom(\mathcal{O}_Y, \mathcal{O}_{Y_K}(h - E)) = \mathbb{C}[-1]$$

so that there is a non trivial extension

$$0 \rightarrow \mathcal{O}_{Y_K}(h - E) \rightarrow \mathbb{L}_{\mathcal{O}_Y} \mathcal{O}_{Y_K}(h - E) \rightarrow \mathcal{O}_Y \rightarrow 0 \quad (148)$$

To compute the \mathcal{U}^\perp component of $\mathbb{L}_{\mathcal{O}_Y} \mathcal{O}_{Y_K}(h - E)$, use the defining sequence (148):

$$0 \rightarrow V/\mathcal{U} \otimes \mathcal{O}_{Y_K}(h - E) \rightarrow V/\mathcal{U} \otimes \mathbb{L}_{\mathcal{O}_Y} \mathcal{O}_{Y_K}(h - E) \rightarrow V/\mathcal{U} \rightarrow 0$$

Note that the connection morphism

$$V \cong H^0(V/\mathcal{U}) \rightarrow H^1(V/\mathcal{U} \otimes \mathcal{O}_{Y_K}(h-E))$$

is isomorphic to the second map in the exact sequence

$$0 \rightarrow H^1(\mathcal{U} \otimes \mathcal{O}_{Y_K}(h-E)) \rightarrow V \otimes H^1(\mathcal{O}_{Y_K}(h-E)) \rightarrow H^1(V/\mathcal{U} \otimes \mathcal{O}_{Y_K}(h-E)) \rightarrow 0$$

induced by the tautological sequence on Y . Let us now prove that the above sequence is isomorphic to

$$0 \rightarrow K \rightarrow V \rightarrow V/K \rightarrow 0$$

For each component e_i of the exceptional locus, get the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_{e_i}) & & \\ \downarrow & \searrow & \\ H^1(\mathcal{O}_{Y_K}(-E)) & \rightarrow & V \otimes H^1(\mathcal{O}_{Y_K}(h-E)) \end{array}$$

The diagonal map is the natural injection of a component of the intersection $\mathbb{P}(K) \cap \sigma(A)$ into V , so that as i varies among all components, the images span $K \subset V$. As $h^1(\mathcal{O}_{Y_K}(-E)) = 3$, we have proved that the mutation of $\mathcal{O}_{Y_K}(h-E)$ across $\mathcal{U}^\perp, \mathcal{O}_Y, \mathcal{O}_Y(H)$ is the cone of

$$\mathcal{O}_{Y_K}(h-E) \rightarrow \{K \otimes \mathcal{U}^\perp \rightarrow \mathcal{O}_Y\}$$

As this last cone is isomorphic to a direct sum of shifts of \mathcal{U} , the computation of the rank yields the following decomposition for $\mathcal{O}_{Y_K}(h-E)$

$$\{\text{Ker}(K \otimes A \rightarrow V^*) \otimes \mathcal{U} \rightarrow K \otimes \mathcal{U}^\perp \rightarrow \mathcal{O}_Y\} \quad (149)$$

Finally, we can use decomposition (149) to show that

$$\text{Ker}(K \otimes A \rightarrow V^*) \cong \text{Ext}^1(\mathcal{O}_{Y_K}(h-E), \mathcal{U})$$

This is clear as the \mathcal{U}^\perp and \mathcal{O} part of (149) do not contribute to $\text{Hom}(-, \mathcal{U})$.

The same isomorphisms, performed in families over $B \setminus D_{tri}$, show that

$$\text{Ker}^{B \setminus D_{tri}} \cong \mathcal{G}^*|_{B \setminus D_{tri}}$$

up to a twist by a line bundle on $B \setminus D_{tri}$. □

Our claim is that the moduli space of non-special instantons of charge 3 is the open subset of $\mathbb{P}_B(\mathcal{G})$ parametrizing surjective maps from \mathcal{U}^* to \mathcal{O}_{S_K} . We will denote it by $\mathbb{P}(\mathcal{G})^\circ$. By lemma 6.32 there are no such maps if $K \in D_{tri}$, so that the image of $\mathbb{P}(\mathcal{G})^\circ$ in B does not intersect D_{tri} . By lemma 6.31 and lemma 6.32, the image of $\mathbb{P}(\mathcal{G})^\circ$ in B is exactly the complement of D_{tri} .

Theorem 6.34. *The moduli space \mathcal{MT}_3^n of non-special instantons of charge 3 is isomorphic to $\mathbb{P}(\mathcal{G})^\circ$.*

Proof. First we prove that $\mathbb{P}(\mathcal{G})^\circ$ has a natural identification with an open subscheme of \mathcal{MT}_3^n . There is a natural map from $\mathbb{P}(\mathcal{G})^\circ$ to \mathcal{MT}_3^n induced by the relative version of lemma 6.27.

The diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{G})^\circ & \longrightarrow & \mathcal{MT}_3^n \\ & \searrow & \downarrow \beta \\ & & B \end{array} \quad (150)$$

is commutative by proposition 6.16. By lemma 6.28 the map is injective on closed points. Moreover the two varieties have the same dimension and are smooth, as explained in remark 4.10 about the GIT construction. It follows from Zariski's main theorem that the map is an open embedding.

Now we prove that the map from $\mathbb{P}(\mathcal{G})^\circ$ to \mathcal{MT}_3^n is surjective by proving that every non-special instanton is the kernel of a surjective map from \mathcal{U}^* to $\Phi(K)$ for some K .

First, note that by proposition 6.11 the diagram (150) factors as

$$\begin{array}{ccc} \mathbb{P}(\mathcal{G})^\circ & \longrightarrow & \mathcal{MT}_3^n \\ & \searrow & \downarrow \beta \\ & & B^n \end{array} \quad (151)$$

Over $Y \times (B^n)$ we can consider the sheaf

$$\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)) \quad (152)$$

and think of it as of a family of Gieseker semistable pure sheaves. The fact that the sheaves $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$ are pure follows from lemma 3.24. Their stability follows from the fact that they have rank 1 on their support (which is irreducible, by 3.3), so that they have no non-trivial saturated subsheaves. It follows that (152) defines a regular map

$$B^n \rightarrow \mathcal{M}_{\text{ch}(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))}$$

into the moduli space of Gieseker semistable pure sheaves with the same Chern character as $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$.

One can also define a family of such sheaves over \mathcal{MT}_3^n by taking the cokernel of the coevaluation map

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{U}^*$$

Note that this family of cokernels restricts to $\mathbb{P}(\mathcal{G})^\circ$ as

$$0 \rightarrow \mathcal{E}|_{\mathbb{P}(\mathcal{G})^\circ} \rightarrow \mathcal{U}^* \rightarrow \Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)) \rightarrow 0$$

where the surjective map is the universal one on $\mathbb{P}(\mathcal{G})^\circ$ from \mathcal{U}^* to $\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h))$.

What we have proved so far is that there is a diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{G})^\circ & \xhookrightarrow{\quad} & \mathcal{MT}_3^n \\ \downarrow & \searrow \beta & \downarrow \text{ck}=\text{coker}(\mathcal{E} \rightarrow \mathcal{U}^*) \\ B^n & \xrightarrow{\delta} & \mathcal{M}_{\text{ch}(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))} \end{array}$$

and that the square in it is commutative. As $\mathbb{P}(\mathcal{G})^\circ$ is an open Zariski dense subvariety of \mathcal{MT}_3^n and as the moduli space $\mathcal{M}_{\text{ch}(\Phi_{\text{Bl}}(\mathcal{O}_{\mathbb{P}(K)}(2h)))}$ is separated, the two maps $\delta \circ \beta$ and ck coincide. In particular, for any non-special instanton E of charge 3 we have an exact sequence

$$0 \rightarrow E \rightarrow \mathcal{U}^* \rightarrow \Phi_{\text{Bl}}\mathcal{O}_{\beta(E)}(2h) \rightarrow 0$$

that is to say that E is in the image of $\mathbb{P}(\mathcal{G})^\circ$. \square

As a corollary, we get the following relation between jumping lines for E and bisecants to $\sigma(A)$ contained in $\mathbb{P}(\beta(E))$.

Corollary 6.35. *Given a bisecant L to $\sigma(A)$ and a non-special instanton E such that $L \subset \beta(E)$, the image $L^+ = p_Y(\tilde{L})$ of the strict transform \tilde{L} of L in $\mathbb{P}_Y(\mathcal{U})$ is a 1-jumping line for E .*

Proof. By theorem 6.34 for any non-special instanton there is an exact sequence

$$0 \rightarrow E \rightarrow \mathcal{U}^* \rightarrow \mathcal{O}_{Y_{\beta(E)}}(2h) \rightarrow 0$$

By twisting it by $\mathcal{O}_Y(-H)$ and restricting it to L^+ we get a long exact sequence

$$0 \rightarrow \text{Tor}_1(\mathcal{O}_{Y_{\beta(E)}}(2h-H), \mathcal{O}_{L^+}) \rightarrow E|_{L^+}(-1) \rightarrow \mathcal{O}_{L^+} \oplus \mathcal{O}_{L^+}(-1) \rightarrow \mathcal{O}_{L^+ \cap Y_{\beta(E)}}(2h-H) \rightarrow 0 \quad (153)$$

As L^+ lies in $Y_{\beta(E)}$ and h restricts to L as $\mathcal{O}_L(1)$, sequence (153) becomes

$$0 \rightarrow \mathcal{O}_{L^+} \rightarrow E|_{L^+}(-1) \rightarrow \mathcal{O}_{L^+} \oplus \mathcal{O}_{L^+}(-1) \rightarrow \mathcal{O}_{L^+}(1) \rightarrow 0 \quad (154)$$

which shows that L^+ is 1-jumping for E \square

Remark 6.36. For $K \notin D_{tri} \cup D_{fat}$ there are 6 bisecants to $\sigma(A)$ contained in $\mathbb{P}(K)$, corresponding to lines in $\mathbb{P}(K)$ via the 4 points $\mathbb{P}(K) \cap \sigma(A)$. If $K \in D_{fat} \setminus D_{tri}$ there will be in general less and less bisecants to $\sigma(A)$, as long as $K \notin \tau(\mathbb{P}(A))$.

When $K \in \tau(\mathbb{P}(A))$, i.e. when $\mathbb{P}(K)$ is the tangent plane to $\sigma(A)$ at some point $a_L \in \mathbb{P}(A)$, there are infinitely many bisecants to $\sigma(A)$ contained in $\mathbb{P}(K)$, namely all lines in $\mathbb{P}(K)$ through a_L . It follows that for instantons E such that $\beta(E) \in \tau(\mathbb{P}(A)) \setminus D_{tri}$ there is a $\mathbb{P}^1 \subset \mathbb{P}(A)$ of jumping lines which is the \mathbb{P}^1 of lines intersecting L .

In other words, corollary 6.35 says that the family of instantons with a fixed $\beta(E)$ induces a linear system of cubic plane curves (the supports of the associated theta characteristics) whose base locus corresponds in the dual $\mathbb{P}(A)$ to the intersection of $\mathbb{P}(K)$ and $\sigma(A)$.

7 An example: degenerate theta-characteristics

In this section we will discuss the possible singularities of the theta-characteristic associated with a charge 3 instanton. We will prove that the instantons whose associated theta-characteristic is not locally free are the special instantons, and we will provide a

geometric description of a family of instantons whose associated theta-characteristic is supported on a reducible curve.

Classically, a theta-characteristic on a smooth curve is a line bundle such that

$$\mathcal{M} \otimes \mathcal{M} \cong \omega_C \quad (155)$$

Unfortunately, condition (155) forces \mathcal{M} to be a line bundle, while a family of smooth curves with theta-characteristic can degenerate flatly to a singular curve with a torsion sheaf on it. On a Gorenstein curve one can still make sense of the condition

$$\mathcal{M} \cong R\mathcal{H}om(\mathcal{M}, \omega_C) \quad (156)$$

which in the smooth case is equivalent to (155).

Definition 7.1. A generalized theta-characteristic on a Gorenstein curve C is a sheaf \mathcal{M} on C such that condition (156) holds.

Generalized theta-characteristics can be not locally free and can have rank greater than one (as condition (156) is for example invariant under direct sums). Given an instanton E and provided that it is trivial on the generic line, theorem 4.14 associates a theta-characteristic θ_E with E . Moreover, all theta-characteristics θ_E arising from instantons satisfy $H^0(\theta_E) = 0$: such generalized theta-characteristics are called non-degenerate.

There are several open questions about how degenerate can θ_E be for an instanton E : can the support of θ_E be singular? Can it be reducible? Can the sheaf θ_E be not locally free? It turns out the answer to all the above questions is positive, as it is proved in the next propositions.

Proposition 7.2. *Let E be an instanton of charge 3, then θ_E is not locally free if and only if E is special.*

Proof. By corollary 6.24 an instanton of charge 3 is special if and only if it has a line L where its order of jump is greater than 1. By proposition 4.16 this is equivalent to θ_E having rank 2 at some point of its support, i.e. to the fact that θ_E is not locally free on its schematic support. \square

Proposition 7.3. *Let E be a non-special instanton such that $\beta(E) \subset \mathbb{P}(V)$ is a tangent plane to $\sigma(A)$. Then the support of θ_E is reducible.*

Proof. First, note that by 4.17 the support of θ_E is the intersection of $\mathbb{P}(A)$ and of the symmetric discriminant cubic fourfold Δ_H inside $\mathbb{P}(S^2 H)$. More precisely, the sheaf θ_E is the pullback of the sheaf of cokernels of a symmetric form in H , which is a sheaf of rank 1 over the generic point of Δ_H and of rank 2 over a surface. If $\mathbb{P}(A) \subset \Delta_H$, then it intersects the locus of corank 2 symmetric forms, so that the rank of θ_E is 2 at least at one point of $\mathbb{P}(A)$. As the instanton is non-special, this is impossible by proposition 6.24, so that the support of θ_E is not the whole $\mathbb{P}(A)$.

By corollary 6.35, all tangent lines to $\sigma(A)$ which are contained in $\beta(E)$ are jumping lines for E . As $\beta(E)$ is a tangent plane to $\sigma(A)$ at some point a , the \mathbb{P}^1 of lines in $\beta(E)$ through a corresponds to a \mathbb{P}^1 of lines in Y all of them intersecting L_a . By proposition 2.26, the family of lines intersecting L_a is a line in $\mathbb{P}(A)$, so that we have shown that the support of θ_E contains a line. \square

Remark 7.4. One can construct explicitly a theta-characteristic coming from an instanton and supported on a reducible curve. This can be done by taking coordinates x, y, z on $\mathbb{P}(A)$ of weight $-2, 2, 0$ with respect to the action of SL_2 . Then the object

$$\left\{ \mathcal{O}_{\mathbb{P}(A)}(-2)^{\oplus 3} \xrightarrow{\Theta} \mathcal{O}_{\mathbb{P}(A)}(-1)^{\oplus 3} \right\} \quad (157)$$

where Θ is given by the matrix

$$\Theta = \begin{pmatrix} 0 & x & y \\ -x & y & z \\ -y & -z & 0 \end{pmatrix}$$

The object (157) is \mathbb{C}^* -equivariant, and this fact allows to prove that the conditions of theorem 4.8 hold for it.

8 A desingularization of $\overline{\mathcal{MI}}_3$

The compactification of \mathcal{MI}_3 which we have constructed in section 6 is singular on a subvariety of codimension 3. We will show it by constructing a desingularization for it, which we will denote by $\widetilde{\mathcal{MI}}_3$, and a small contraction from $\widetilde{\mathcal{MI}}_3$ to $\overline{\mathcal{MI}}_3$.

There is a commutative diagram

$$\begin{array}{ccc} \mathbb{F}l_B(3, 4, \mathcal{K} \otimes A) & \xrightarrow{\pi_3} & \mathbb{G}r_B(4, \mathcal{K} \otimes A) \\ \pi_4 \downarrow & & \downarrow \pi_4 \\ \mathbb{G}r_B(3, \mathcal{K} \otimes A) & \xrightarrow{\pi_3} & B \end{array} \quad (158)$$

On the relative Grassmannians there are relative tautological injections

$$\mathcal{T}_3^{\text{rel}} \rightarrow \pi_3^* \mathcal{K} \otimes A \quad (159)$$

and

$$\mathcal{T}_4^{\text{rel}} \rightarrow \pi_4^* \mathcal{K} \otimes A \quad (160)$$

The bundle $\mathcal{K} \otimes A$ is a subbundle of $V \otimes A \otimes \mathcal{O}_B$, so that the natural evaluation of A on V induces a map

$$\mathcal{K} \otimes A \rightarrow V^* \otimes \mathcal{O}_B \quad (161)$$

extending to an exact sequence

$$0 \rightarrow \mathcal{K}er \rightarrow \mathcal{K} \otimes A \rightarrow V^* \otimes \mathcal{O}_B \rightarrow \kappa_* \mathcal{O}_{\mathbb{P}(A)}(2) \rightarrow 0$$

which is discussed in proposition 3.6.

Lemma 8.1. *There is a unique SL_2 -equivariant line subbundle*

$$\mathcal{O}_B(-1) \rightarrow A \otimes \mathcal{K} \quad (162)$$

and it factors via $\mathcal{K} \rightarrow A \otimes \mathcal{K}$.

Proof. Note that $\mathcal{K}(1)$ is isomorphic to $\Lambda^2 \mathcal{K}^*$. The space of sections of $\Lambda^2 \mathcal{K}^*$ is $\Lambda^2 V^*$ and it contains a unique SL_2 -invariant subspace of dimension 3. It follows that there is a unique SL_2 -invariant choice of the map (162).

In order to check that it factors via

$$\mathcal{K} \rightarrow A \otimes \mathcal{K}$$

we check that the composition

$$\mathcal{O}_B(-1) \rightarrow A \otimes \mathcal{K} \rightarrow V^* \otimes \mathcal{O}_B$$

vanishes. This last fact holds because there are no non-zero SL_2 -invariant elements in $V^* \otimes \Lambda^2 V$. \square

Denote by \mathcal{F} the rank 8 vector bundle which is the quotient (162). There is a natural embedding of relative Grassmannians

$$\mathrm{Gr}_B(3, \mathcal{F}) \rightarrow \mathrm{Gr}_B(4, A \otimes \mathcal{K})$$

and a tautological embedding

$$\mathcal{T}_{\mathcal{F}}^{\mathrm{rel}} \rightarrow \pi_{\mathcal{F}}^* \mathcal{F}$$

where $\mathcal{T}_{\mathcal{F}}^{\mathrm{rel}}$ is the rank 3 tautological vector bundle and $\pi_{\mathcal{F}}$ is the canonical projection to B .

By lemma 8.1 the map (161) factors via the natural projection from $A \otimes \mathcal{K}$ to \mathcal{F} . Denote by $\sigma_{\mathcal{F}}$ the composition

$$\sigma_{\mathcal{F}} : \mathcal{T}_{\mathcal{F}}^{\mathrm{rel}} \rightarrow \pi_{\mathcal{F}}^* \mathcal{F} \rightarrow V^* \otimes \mathcal{O}_B \quad (163)$$

and its zero locus by $Z_{\mathcal{F}}$. Analogously there are maps

$$\begin{aligned} \sigma_3 : \mathcal{T}_3^{\mathrm{rel}} &\rightarrow V^* \otimes \mathcal{O} \\ \sigma_4 : \mathcal{T}_4^{\mathrm{rel}} &\rightarrow V^* \otimes \mathcal{O} \end{aligned}$$

over $\mathrm{Gr}_B(3, \mathcal{K} \otimes A)$ and $\mathrm{Gr}_B(4, \mathcal{K} \otimes A)$, whose zero loci will be denoted by Z_3 and Z_4 . By its definition, $\overline{\mathcal{MT}}_3$ is the zero locus Z_3 of σ_3 . Moreover, we will denote by $\tilde{Z}_{\mathcal{F}}, \tilde{Z}_4$ the pullbacks of $Z_{\mathcal{F}}, Z_4$ via $\tilde{\pi}_3$.

Lemma 8.2. *The section $\sigma_{\mathcal{F}}$ is regular. Its zero locus $Z_{\mathcal{F}}$ is irreducible and is isomorphic to $\mathrm{Bl}_{B^s} B$. The intersection of $\pi_4^{-1}(B^s)$ and $Z_{\mathcal{F}}$ in $\mathrm{Gr}_B(4, A \otimes \mathcal{K})$ embeds into $\pi_4^{-1}(B^s)$ as*

$$\mathrm{Gr}_{B^s}(3, \mathcal{K} \otimes \mathcal{O}_{B^s}(-3)) \subset \mathrm{Gr}_{B^s}(4, A \otimes \mathcal{K})$$

and into $Z_{\mathcal{F}}$ as

$$E \subset \mathrm{Bl}_{B^s} B$$

where E is the exceptional divisor of the blow up.

Proof. First, the zero locus $Z_{\mathcal{F}}$ has a stratification into $Z_{\mathcal{F}}^s$ and $Z_{\mathcal{F}}^n$ induced by pulling back the stratification B^s, B^n . Note that $Z_{\mathcal{F}}^n$ is isomorphic to B^n , as the kernel of

$$\pi_{\mathcal{F}}^* \mathcal{F} \rightarrow V^* \otimes \mathcal{O}_B \quad (164)$$

is locally free of rank 3 over B^n . On the other hand, for the same reason, $Z_{\mathcal{F}}^n$ is a \mathbb{P}^3 fibration over B^s , so that its dimension is 5. It follows that $Z_{\mathcal{F}}$ is 6-dimensional inside a 21-dimensional space where it is cut by a section $\sigma_{\mathcal{F}}$ of a rank 15 vector bundle. As the ambient space is smooth, the section is regular and $Z_{\mathcal{F}}$ is irreducible.

Next, let us construct a map from $Z_{\mathcal{F}}$ to the blow up $\text{Bl}_{B^s} B$. Note that by the definition of $Z_{\mathcal{F}}$ the map (163) vanishes over $Z_{\mathcal{F}}$, so that there is an induced

$$\mathcal{F}/\mathcal{T}_{\mathcal{F}}^{\text{rel}} \rightarrow V^* \otimes \mathcal{O}_{\text{Gr}_{\mathcal{F}}} \quad (165)$$

By proposition 3.6 its cokernel is a line bundle supported on $\pi_{\mathcal{F}}^{-1}(B^s)$, so that the rank of (165) drops at most by 1. It follows that the schematic support of the cokernel of (165) is cut by the determinant of (165). This shows that $\pi_{\mathcal{F}}^{-1}(B^s)$ is a Cartier divisor and by universal property induces a unique map to the blow up of B in B^s .

In the other direction, again by proposition 3.6 and by definition of blow up, the cokernel of the pullback to $\text{Bl}_{B^s} B$ of (161) is a line bundle supported on a Cartier divisor. It follows by Tor-dimension that the kernel of the pullback of (161) is a rank 4 subbundle of $A \otimes \mathcal{K}$. Moreover, by lemma 8.1, the line subbundle

$$\pi_4^* \mathcal{O}_B(-1) \rightarrow A \otimes \mathcal{K}$$

factors via this kernel, inducing a rank 3 subbundle of \mathcal{F} such that the composition with

$$\pi_{\mathcal{F}}^* \mathcal{F} \rightarrow V^* \otimes \mathcal{O}_B$$

vanishes. By universal property of $Z_{\mathcal{F}}$ this induces a map from $\text{Bl}_{B^s} B$ to it.

Moreover, by their construction, these two maps are relative to B and are isomorphisms out of the preimage of B^s . Note that there is a unique map from $\text{Bl}_{B^s} B$ to itself which is relative to B and it is the identity. It follows that

$$\text{Bl}_{B^s} B \rightarrow Z_{\mathcal{F}} \quad (166)$$

is a closed embedding. Note that $Z_{\mathcal{F}}$ is generically reduced and that it cannot have embedded components as it is cut by a regular section. It follows that $Z_{\mathcal{F}}$ is reduced, so that (166) is a closed embedding of irreducible reduced varieties of the same dimension. This is enough to conclude that it is an isomorphism.

As for the embedding statements, they follow directly from the definitions of $Z_{\mathcal{F}}$ and of Bl_{B^s} . \square

Lemma 8.3. *The section σ_4 is regular and its zero locus Z_4 has 2 components: $Z_{\mathcal{F}}$ and*

$$Z_4^v := \text{Gr}_{B^s}(3, \text{Ker}^s)$$

Proof. As in lemma 8.2 one can check that σ_4 is regular. Clearly there is an inclusion of $Z_{\mathcal{F}}$ in Z_4 which is an isomorphism out of the preimage of B^s . On the other hand the restriction of Z_4 to B^s is by definition $\mathrm{Gr}_{B^s}(3, \mathcal{K}er^s)$. \square

Recall that $\tilde{Z}_{\mathcal{F}}$ is the result of the fiber product

$$\begin{array}{ccc} \tilde{Z}_{\mathcal{F}} & \xrightarrow{\tilde{\pi}_3} & Z_{\mathcal{F}} \\ \tilde{\pi}_4 \downarrow & & \downarrow \pi_4 \\ \overline{\mathcal{M}\mathcal{I}}_3 & \xrightarrow{\pi_3} & B \end{array} \quad (167)$$

where we abuse the notation of (158) and use the same letters for the restricted projections.

Our next claim is that $\overline{\mathcal{M}\mathcal{I}}_3$ is singular on a 6-dimensional locus and that $\tilde{Z}_{\mathcal{F}}$ is a small resolution of singularities for it. We also show that $\tilde{Z}_{\mathcal{F}}$ is isomorphic to the blow up of $\overline{\mathcal{M}\mathcal{I}}_3$ in the (non-Cartier) divisor $\overline{\mathcal{M}\mathcal{I}}_3^s$, which is the closure of $\mathcal{M}\mathcal{I}_3^s$ inside $\overline{\mathcal{M}\mathcal{I}}_3$.

Theorem 8.4. *There is an isomorphism*

$$\phi : \tilde{Z}_{\mathcal{F}} \rightarrow \mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}}_3^s} \overline{\mathcal{M}\mathcal{I}}_3$$

fitting in a commutative diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}\mathcal{I}}_3 := \tilde{Z}_{\mathcal{F}} & \xrightarrow{\tilde{\pi}_3} & \mathrm{Bl}_{B^s} B & & \\ & \searrow \phi & \nearrow * & & \\ & & \mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}}_3^s} \overline{\mathcal{M}\mathcal{I}}_3 & & \\ \tilde{\pi}_4 \downarrow & & \swarrow & \searrow & \downarrow \pi_4 = b \\ \overline{\mathcal{M}\mathcal{I}}_3 & \xrightarrow{\pi_3 = \beta} & B & & \end{array} \quad (168)$$

The singular locus of $\overline{\mathcal{M}\mathcal{I}}_3$ is

$$\mathrm{Gr}_{B^s}(2, \mathcal{K}er^s / \mathcal{O}_{B^s}(-3)) \subset \mathrm{Gr}_{B^s}(3, \mathcal{K}er^s) \cong \overline{\mathcal{M}\mathcal{I}}_3^s$$

and the map $\tilde{\pi}_4$ is a small resolution of singularities for $\overline{\mathcal{M}\mathcal{I}}_3$ such that

$$\mathcal{O}_{\overline{\mathcal{M}\mathcal{I}}_3} \rightarrow R\tilde{\pi}_{4*} \mathcal{O}_{\tilde{Z}_{\mathcal{F}}}$$

is an isomorphism, i.e. the singularity of $\overline{\mathcal{M}\mathcal{I}}_3$ is rational.

Proof. First, we construct the map

$$\mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}}_3^s} \overline{\mathcal{M}\mathcal{I}}_3 \rightarrow \mathrm{Bl}_{B^s} B$$

By universal property of the blow-up, given a map from $\mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}}_3^s} \overline{\mathcal{M}\mathcal{I}}_3$ to B it is enough to show that the pullback of B^s is a Cartier divisor. As the rightmost square is cartesian

by proposition 6.15 and as the composition of cartesian squares is cartesian, the outer square of

$$\begin{array}{ccccc}
\mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}_3}} \overline{\mathcal{M}\mathcal{I}_3} \times_{\overline{\mathcal{M}\mathcal{I}_3}} \overline{\mathcal{M}\mathcal{I}_3}^s & \longrightarrow & \overline{\mathcal{M}\mathcal{I}_3}^s & \longrightarrow & \mathbb{P}(A) \\
\downarrow & \square & \downarrow & \square & \downarrow \kappa \\
\mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}_3}^s} \overline{\mathcal{M}\mathcal{I}_3} & \longrightarrow & \overline{\mathcal{M}\mathcal{I}_3} & \longrightarrow & B
\end{array} \tag{169}$$

is cartesian. Moreover, by definition of the blow up of $\overline{\mathcal{M}\mathcal{I}_3}^s \subset \overline{\mathcal{M}\mathcal{I}_3}$, the leftmost vertical arrow is the embedding of the exceptional divisor, which is always a Cartier divisor.

Now we construct the map

$$\phi : \tilde{Z}_{\mathcal{F}} \rightarrow \mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}_3}} \overline{\mathcal{M}\mathcal{I}_3}$$

Again, by universal property of blow up we want to check that the leftmost vertical arrow in

$$\begin{array}{ccccc}
\tilde{Z}_{\mathcal{F}} \times_{\overline{\mathcal{M}\mathcal{I}_3}} \overline{\mathcal{M}\mathcal{I}_3}^s & \longrightarrow & \overline{\mathcal{M}\mathcal{I}_3}^s & \longrightarrow & \mathbb{P}(A) \\
\downarrow & \square & \downarrow & \square & \downarrow \kappa \\
\tilde{Z}_{\mathcal{F}} & \xrightarrow{\tilde{\pi}_4} & \overline{\mathcal{M}\mathcal{I}_3} & \xrightarrow{\beta} & B
\end{array}$$

is the embedding of a Cartier divisor. By commutativity of square (168), this is equivalent to the fact that the leftmost vertical arrow in

$$\begin{array}{ccccc}
\tilde{Z}_{\mathcal{F}} \times_{\mathrm{Bl}_{B^s} B} E & \longrightarrow & E & \longrightarrow & \mathbb{P}(A) \\
\downarrow & \square & \downarrow & \square & \downarrow \kappa \\
\tilde{Z}_{\mathcal{F}} & \xrightarrow{\tilde{\pi}_3} & \mathrm{Bl}_{B^s} B & \xrightarrow{b} & B
\end{array}$$

is the embedding of a Cartier divisor. This last fact is true as E is a Cartier divisor in $\mathrm{Bl}_{B^s} B$ and as $\tilde{\pi}_3$ is a dominant map from an irreducible scheme.

Next, we prove commutativity of all triangles in (168). The triangles which are not marked with $*$ are commutative by construction. The triangle marked with $*$ is commutative because by universal property of the blow up there is a unique map from $\tilde{Z}_{\mathcal{F}}$ to $\mathrm{Bl}_{B^s} B$ which commutes with the projections to B , and both $\tilde{\pi}_3$ and

$$\tilde{Z}_{\mathcal{F}} \xrightarrow{\phi} \mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}_3}} \overline{\mathcal{M}\mathcal{I}_3} \rightarrow \mathrm{Bl}_{B^s} B$$

commute with the projections $\pi_3 \circ \tilde{\pi}_4$ and π_4 .

We still have to prove that ϕ is an isomorphism. As $\overline{\mathcal{M}\mathcal{I}_3}$ is integral, also its blow-up in $\overline{\mathcal{M}\mathcal{I}_3}^s$ is integral. As both $\tilde{Z}_{\mathcal{F}}$ and $\mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}_3}} \overline{\mathcal{M}\mathcal{I}_3}$ are integral proper schemes of dimension 9, so that in order to check that ϕ is an isomorphism it is enough to check that it is a closed embedding. By their definition, the maps $\tilde{\pi}_3$ and $\tilde{\pi}_4$ induce a closed embedding

$$\tilde{Z}_{\mathcal{F}} \xrightarrow{(\tilde{\pi}_3, \tilde{\pi}_4)} \mathrm{Gr}(3, \mathcal{K} \otimes A) \times_B \mathrm{Gr}(4, \mathcal{K} \otimes A)$$

As we have just proved that diagram (168) is commutative, there is an induced map from $\mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}}_3} \overline{\mathcal{M}\mathcal{I}}_3$ to the fiber product of π_3 and π_4 such that the triangle

$$\begin{array}{ccc} \widetilde{Z}_{\mathcal{F}} & & \\ \downarrow \phi & \searrow & \\ \mathrm{Bl}_{\overline{\mathcal{M}\mathcal{I}}_3} \overline{\mathcal{M}\mathcal{I}}_3 & \rightarrow & \mathrm{Gr}(3, \mathcal{K} \otimes A) \times_B \mathrm{Gr}(4, \mathcal{K} \otimes A) \end{array}$$

commutes. As ϕ is the first factor of a closed embedding, it is itself a closed embedding, and therefore an isomorphism.

As for the last part of the statement, $\widetilde{\pi}_3$ is a smooth map as it is the projection associated with the projectivization of a vector bundle, so that $\widetilde{Z}_{\mathcal{F}}$ is smooth as $\mathrm{Bl}_{B^s} B$ is smooth.

To describe the singular locus of $\overline{\mathcal{M}\mathcal{I}}_3$, recall that by lemma 8.1 there is an SL_2 -equivariant exact sequence

$$0 \rightarrow \mathcal{O}_B(-1) \rightarrow A \otimes \mathcal{K} \rightarrow \mathcal{F} \rightarrow 0$$

inducing an embedding

$$\mathrm{Gr}_{B^s}(2, \mathcal{K} \mathrm{er}^s / \mathcal{O}_{B^s}(-3)) \rightarrow \mathrm{Gr}_{B^s}(3, \mathcal{K} \mathrm{er}^s) \cong \overline{\mathcal{M}\mathcal{I}}_3^s$$

Over points in $\mathrm{Gr}_{B^s}(2, \mathcal{K} \mathrm{er}^s / \mathcal{O}_{B^s}(-3))$ the fiber of $\widetilde{\pi}_4$ is \mathbb{P}^1 , while over the complement of $\mathrm{Gr}_{B^s}(2, \mathcal{K} \mathrm{er}^s / \mathcal{O}(-3))$ inside $\overline{\mathcal{M}\mathcal{I}}_3$ it has relative dimension 0.

As a consequence, each point in $\mathrm{Gr}_{B^s}(2, \mathcal{K} \mathrm{er}^s / \mathcal{O}(-3))$ is in the singular locus, as otherwise, by choosing a small enough neighborhood of the point, one would find a non-trivial small contraction between smooth varieties, which is impossible.

Now we are going to check that over the complement of $\mathrm{Gr}_{B^s}(2, \mathcal{K} \mathrm{er}^s / \mathcal{O}(-3))$ in $\overline{\mathcal{M}\mathcal{I}}_3$ the map $\widetilde{\pi}_4$ is an isomorphism. Note that the restriction of

$$\widetilde{\pi}_4 : \widetilde{\mathcal{M}\mathcal{I}}_3 \rightarrow \overline{\mathcal{M}\mathcal{I}}_3$$

to

$$\overline{\mathcal{M}\mathcal{I}}_3 \setminus \mathrm{Gr}_{B^s}(2, \mathcal{K} \mathrm{er}^s / \mathcal{O}(-3))$$

is proper of relative dimension 0. If we prove that

$$\mathcal{O}_{\overline{\mathcal{M}\mathcal{I}}_3} \cong R\widetilde{\pi}_{4*} \mathcal{O}_{\widetilde{\mathcal{M}\mathcal{I}}_3}$$

then by lemma 1.35 we have proved that the restricted projection $\widetilde{\pi}_4$ is an isomorphism. This will prove at the same time that the singularity of $\overline{\mathcal{M}\mathcal{I}}_3$ is rational and is the content of lemma 8.5 and lemma 8.6. \square

Lemma 8.5. *The adjunction map*

$$\mathcal{O}_{Z_3} \rightarrow R\widetilde{\pi}_{4*} \mathcal{O}_{\widetilde{Z}_4} \tag{170}$$

is an isomorphism.

Proof. Take the locally free resolution of \mathcal{O}_{Z_4} given by the Koszul complex of

$$V \otimes \mathcal{T}_4^{\text{rel}} \xrightarrow{\sigma_4} \mathcal{O}_{\text{Gr}_B(4, \mathcal{K} \otimes A)} \quad (171)$$

Pull it back via the flat map $\tilde{\pi}_3$ to find a Koszul resolution for $\mathcal{O}_{\tilde{Z}_4}$.

We are now going to pushforward the Koszul resolution for $\mathcal{O}_{\tilde{Z}_4}$ via $\tilde{\pi}_4$. Clearly,

$$R\tilde{\pi}_{4*} \mathcal{O}_{\mathbb{F}l_B(3, 4, A \otimes \mathcal{K})} \cong \mathcal{O}_{\text{Gr}_B(3, A \otimes \mathcal{K})}$$

The tautological flag fits in an exact sequence

$$0 \rightarrow \tilde{\pi}_4^* \mathcal{T}_3^{\text{rel}} \rightarrow \tilde{\pi}_3^* \mathcal{T}_4^{\text{rel}} \rightarrow \mathcal{O}_{\tilde{\pi}_4}(-1) \rightarrow 0 \quad (172)$$

so that

$$R\tilde{\pi}_{4*} \tilde{\pi}_3^* \mathcal{T}_4^{\text{rel}} \cong R\tilde{\pi}_{4*} \tilde{\pi}_4^* \mathcal{T}_3^{\text{rel}}$$

By projection formula

$$R\tilde{\pi}_{4*} \tilde{\pi}_4^* \mathcal{T}_3^{\text{rel}} \cong \mathcal{T}_3^{\text{rel}}$$

As the projections $\tilde{\pi}_4$ and $\tilde{\pi}_3$ are SL_2 -equivariant, the SL_2 -invariant section (171) is transformed into the unique SL_2 -invariant section

$$V \otimes \mathcal{T}_3^{\text{rel}} \xrightarrow{\sigma_3} \mathcal{O}_{\text{Gr}_B(3, \mathcal{K} \otimes A)}$$

In order to prove (170), it is enough to prove that

$$R^i \tilde{\pi}_{4*} \tilde{\pi}_3^* \Lambda^i(\mathcal{T}_4^{\text{rel}} \otimes V)$$

vanish for $i > 0$. The exterior powers

$$\Lambda^i(\mathcal{T}_4^{\text{rel}} \otimes V)$$

admit a filtration (see e.g. [Wey03]) with subquotients

$$\Sigma^\alpha \mathcal{T}_4^{\text{rel}} \otimes \Sigma^{\alpha^T} V \quad (173)$$

where Σ^α is the Schur functor associated with a Young diagram α such that $|\alpha| = i$. Note that (173) is not zero only when α is contained in a 5-by-4 rectangle, as the ranks of $\mathcal{T}_4^{\text{rel}}$ and V are respectively 4 and 5.

The Schur powers of $\mathcal{T}_4^{\text{rel}}$ can be rewritten in terms of Schur powers of $\mathcal{T}_3^{\text{rel}}$ and $\mathcal{O}_{\tilde{\pi}_4}(-1)$. This can be done via a natural filtration for

$$\Sigma^\alpha \mathcal{T}_4^{\text{rel}}$$

induced by the tautological flag (172). The associated graded object of this natural filtration is

$$\bigoplus_{\beta \subseteq \alpha} \left(\Sigma^\beta \mathcal{T}_3^{\text{rel}} \otimes \Sigma^{\alpha/\beta} \mathcal{O}_{\tilde{\pi}_4}(-1) \right) \quad (174)$$

where $\Sigma^{\alpha/\beta}$ are the skew Schur functions. As α is contained in a 5-by-4 rectangle

$$\Sigma^{\alpha/\beta}(\mathcal{O}_{\tilde{\pi}_4}(-1)) = \mathcal{O}_{\tilde{\pi}_4}(-j) \quad (175)$$

for $j \in [0, 5]$. As $\tilde{\pi}_4$ is the projectivization of a rank 6 vector bundle, the line bundles have vanishing higher pushforward. Finally, by projection formula the pushforward of summands of (174) is

$$\Sigma^\beta \mathcal{T}_3^{\text{rel}} \otimes R\tilde{\pi}_{4*} \mathcal{O}_{\tilde{\pi}_4}(-j)$$

which is always concentrated in degree 0. \square

Lemma 8.6. *The adjunction map*

$$\mathcal{O}_{\overline{\mathcal{MT}}_3} \rightarrow R\tilde{\pi}_{4*} \mathcal{O}_{\tilde{Z}_{\mathcal{F}}} \quad (176)$$

is an isomorphism.

Proof. On \tilde{Z}_4 there are exact sequences

$$0 \rightarrow \mathcal{I}_{\tilde{Z}_{\mathcal{F}}/\tilde{Z}_4} \rightarrow \mathcal{O}_{\tilde{Z}_4} \rightarrow \mathcal{O}_{\tilde{Z}_{\mathcal{F}}} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{I}_{\tilde{Z}_4^v \cap \tilde{Z}_{\mathcal{F}}/\tilde{Z}_4^v} \rightarrow \mathcal{O}_{\tilde{Z}_4^v} \rightarrow \mathcal{O}_{\tilde{Z}_4^v \cap \tilde{Z}_{\mathcal{F}}} \rightarrow 0$$

By lemma 8.5

$$R\tilde{\pi}_{4*} \mathcal{O}_{\tilde{Z}_4} \cong \mathcal{O}_{Z_3}$$

so that our claim becomes that $R\tilde{\pi}_{4*} \mathcal{I}_{\tilde{Z}_{\mathcal{F}}/\tilde{Z}_4}$ vanishes, or equivalently

$$R\tilde{\pi}_{4*} \mathcal{I}_{\tilde{Z}_4^v \cap \tilde{Z}_{\mathcal{F}}/\tilde{Z}_4^v} = 0 \quad (177)$$

We are now going to prove that (177) is an isomorphism. To this purpose, recall that the intersection of Z_4^v and $Z_{\mathcal{F}}$ inside $\mathbb{G}r_{B^s}(4, \mathcal{K}er^s)$ is described in lemma 8.2 and 8.3. As a consequence, the commutative diagram

$$\begin{array}{ccc} \tilde{Z}_4^v \cap \tilde{Z}_{\mathcal{F}} & \hookrightarrow & \tilde{Z}_4^v \\ & \searrow & \downarrow \\ & & \mathcal{MT}_3^s \end{array}$$

is isomorphic to the commutative diagram

$$\begin{array}{ccc} \mathbb{Fl}_{B^s}(3, \mathcal{O}_{B^s}(-3) \subset 4, \mathcal{K}er^s) & \hookrightarrow & \mathbb{Fl}_{B^s}(3, 4, \mathcal{K}er^s) \\ & \searrow & \downarrow \\ & & \mathbb{G}r_{B^s}(3, \mathcal{K}er^s) \end{array} \quad (178)$$

where the projections are the natural ones to $\mathbb{G}r_{B^s}(3, \mathcal{K}er^s)$ and where the upper left corner is the flag variety of subbundles of $\mathcal{K}er^s$ such that the injection (162) restricted to B^s factors via the 4-dimensional subbundle.

The ideal of the top left corner of diagram (178) is the pullback of $\mathcal{O}(-1)$ from $\mathbb{G}r(4, \mathcal{K}er^s)$ (possibly with a twist by a line bundle on B^s). It follows that it is acyclic on the fibers of the projection to $\mathbb{G}r_{B^s}(3, \mathcal{K}er^s)$, so that (177) holds. \square

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